Differential forms locally:

$\varphi : D \longrightarrow M$ surface

$D \subset \mathbb{R}^2$ open domain

(u,v) coordinates on D.

at any point $p \in D$ $\quad T_p D = \mathbb{R}^2$.

canonical basis : $\{(1,0),(0,1)\}$.

$\varphi : D \xrightarrow{\cong} U \subset M$
diffeomorphism.

du, dv basis of differential 1-forms on U "$=$" D.

So locally, any differential 1-form on U is of the form

$$f\,du + g\,dv$$

where f, g are differentiable functions on U : $f(u,v)$, $g(u,v)$

$$df = \frac{\partial f}{\partial u}\,du + \frac{\partial f}{\partial v}\,dv.$$

$\omega = f\,du + g\,dv \quad$ a 1-form on U.

by definition:

$\cancel{\omega(w_1, w_2) = \partial}$

$$(d\omega)(\varphi_u, \varphi_v) = \frac{\partial}{\partial u}\omega(\varphi_v) - \frac{\partial}{\partial v}\omega(\varphi_u)$$

under the identification

$$\varphi : D \xrightarrow{\cong} U$$

$$T_? D \ni (1,0) \longleftrightarrow \varphi_u \in T_? U$$

$$\text{“}\frac{\partial}{\partial u}\text{”} = (1,0) \qquad du\left(\frac{\partial}{\partial u}\right) = 1$$

$$du\left(\frac{\partial}{\partial v}\right) = 0$$

$$\text{“}\frac{\partial}{\partial v}\text{” “=” } (0,1) \longleftrightarrow \varphi_v$$

$$\omega = f\,du + g\,dv.$$

$$\omega(\varphi_u) = f\,du(\varphi_u) + g\,dv(\varphi_u)$$
$$= f\,du\left(\frac{\partial}{\partial u}\right) + g\,dv\left(\frac{\partial}{\partial u}\right)$$
$$= f$$

$$\omega(\varphi_v) = f\,du\left(\frac{\partial}{\partial v}\right) + g\,dv\left(\frac{\partial}{\partial v}\right)$$
$$= g$$

$$d\omega(\varphi_u, \varphi_v) = d\omega\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$$

$$= \frac{\partial}{\partial u} g - \frac{\partial}{\partial v} f$$

every 2-form can be written as

$$h \, du \wedge dv \qquad h \text{ diff. on } U$$

$$h(du \wedge dv)(\varphi_u, \varphi_v) = h \, du \wedge dv\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$$

$$= h \, du\left(\frac{\partial}{\partial u}\right) \cdot dv\left(\frac{\partial}{\partial v}\right) - du\left(\frac{\partial}{\partial v}\right) dv\left(\frac{\partial}{\partial u}\right)$$

$$= h$$

So $d\omega = \left(\frac{\partial}{\partial u} g - \frac{\partial}{\partial v} f\right) du \wedge dv.$

$$= \left(\frac{\partial g}{\partial u} du + \frac{\partial g}{\partial v} dv\right) \wedge dv$$

$$+ \left(\frac{\partial f}{\partial u} du + \frac{\partial f}{\partial v} dv\right) \wedge du$$

$$= dg \wedge dv + df \wedge du$$

$$= df \wedge du + dg \wedge dv$$

On $\mathbb{R}^3$: coordinates x, y, z.

f 0-forms

$f\,dx + g\,dy + h\,dz$ 1-forms

$f\,dx\wedge dy + g\,dx\wedge dz + h\,dy\wedge dz$ 2-forms.

$f\,dx\wedge dy\wedge dz$ 3-forms.

If $\omega = f\,dx\wedge dy + g\,dx\wedge dz + h\,dy\wedge dz$

then $d\omega = df\wedge dx\wedge dy + dg\wedge dx\wedge dz$
$+ dh\wedge dy\wedge dz.$

$$= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)\wedge dx\wedge dy$$

$+ \;\cdots\cdots$

$$= \frac{\partial f}{\partial z}dx\wedge dy\wedge dz - \frac{\partial g}{\partial y}dx\wedge dy\wedge dz$$
$$+ \frac{\partial h}{\partial x}dx\wedge dy\wedge dz.$$

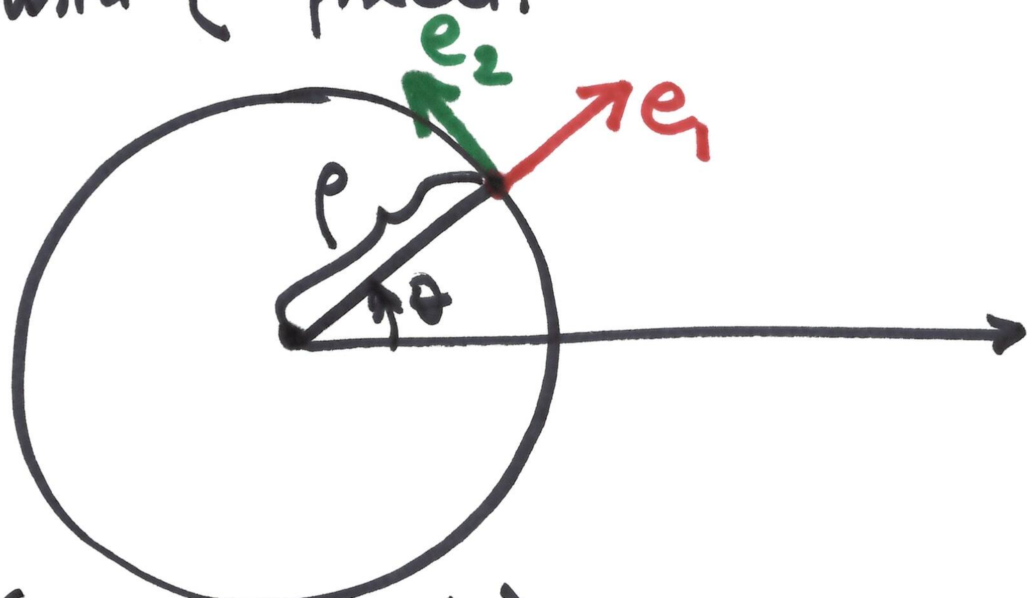
$$= \left(\frac{\partial f}{\partial z} - \frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\right)dx\wedge dy\wedge dz.$$

Back to frame fields:

Circular (or polar) frame field on $\mathbb{R}^2$:

e_1 : direction in which θ is fixed and ρ increases.

e_2 : direction in which θ increases with ρ fixed.



$$e_1 = (\cos\theta, \sin\theta)$$

$$e_2 = (-\sin\theta, \cos\theta).$$

At a point p with polar coordinates (ρ, θ),

$$e_1(p) = (\cos\theta, \sin\theta), e_2(p) = (-\sin\theta, \cos\theta)$$

$$e_1(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

$$e_2(x,y) = \left(-\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}\right)$$

$\{\theta_1, \theta_2\}$ the dual basis for differential 1-forms.

<u>First way</u>: $\theta_1 = f_{11}\,dx + f_{12}\,dy$

$$\theta_2 = f_{21}\,dx + f_{22}\,dy$$

$$\theta_1(e_1) = \theta_1\left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right)$$

$$= f_{11}\cos\theta + f_{12}\sin\theta \equiv 1$$

$$\theta_1(e_2) = \theta_1\left(-\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}\right)$$

$$= -f_{11}\sin\theta + f_{12}\cos\theta \equiv 0$$

Solve the two equations to get:

$$f_{11} = \cos\theta \qquad f_{12} = \sin\theta$$

$$\theta_1 = \cos\theta\,dx + \sin\theta\,dy$$

Similarly: $\theta_2 = -\sin\theta\,dx + \cos\theta\,dy$

$$e_1 \stackrel{?}{=} \frac{\partial}{\partial \rho} \qquad \theta_1 \stackrel{?}{=} d\rho$$

$$e_2 \stackrel{?}{=} \frac{\partial}{\partial \theta} \qquad \theta_2 \stackrel{?}{=} d\theta$$

$$\rho = \sqrt{x^2+y^2} \qquad \rho^2 = x^2+y^2$$

$$\rho\, d\rho = x\,dx + y\,dy$$

$$d\rho = \frac{1}{\rho}(x\,dx + y\,dy)$$

$$= \cos\theta\, dx + \sin\theta\, dy = \boxed{\theta_1 = d\rho}$$

$$\theta = \arctan\left(\frac{y}{x}\right) \text{ or } \tan\theta = \frac{y}{x}$$

$$x\sin\theta = y\cos\theta$$

$$\sin\theta\, dx + x\cos\theta\, d\theta = \cos\theta\, dy - y\sin\theta\, d\theta$$

$$(x\cos\theta + y\sin\theta)\, d\theta = -\sin\theta\, dx + \cos\theta\, dy$$

$$\boxed{\rho\, d\theta = \theta_2}$$

$$\theta_1 \wedge \theta_2 = \rho\, d\rho \wedge d\theta.$$

<u>Form field</u>: $\nabla_v e_1 = \omega_{12}(v)\, e_2$

$$\nabla_v e_2 = -\omega_{12}(v) e_1$$

$$\frac{\partial}{\partial x} e_1 = \omega_{12}\left(\frac{\partial}{\partial x}\right) e_2$$

$$\frac{\partial}{\partial y} e_1 = \omega_{12}\left(\frac{\partial}{\partial y}\right) e_2$$

$$\frac{\partial}{\partial x}(\cos\theta, \sin\theta) = \omega_{12}\left(\frac{\partial}{\partial x}\right)\begin{pmatrix} -\sin\theta, \\ \cos\theta \end{pmatrix}$$

$$\cancel{\left(-\sin\theta \frac{\partial\theta}{\partial x}, \cos\theta \frac{\partial\theta}{\partial x}\right) =}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\theta_1 = d\rho \longrightarrow e_1 = \frac{\partial}{\partial\rho}$$

$$\theta_2 = \rho\, d\theta \longrightarrow e_2 = \frac{1}{\rho}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial\theta} e_1 = \omega_{12}\left(\frac{\partial}{\partial\theta}\right) e_2$$

$$(-\sin\theta, \cos\theta) = \omega_{12}\left(\frac{\partial}{\partial\theta}\right)(-\sin\theta, \cos\theta)$$

$$\Rightarrow \omega_{12}\left(\frac{\partial}{\partial\theta}\right) = 1 \Rightarrow \omega_{12}\left(\frac{1}{\rho}\frac{\partial}{\partial\theta}\right) = \frac{1}{\rho}$$

$$\frac{\partial}{\partial \rho} e_1 = \omega_{12}\left(\frac{\partial}{\partial \rho}\right) e_2$$

$$0 = \omega_{12}\left(\frac{\partial}{\partial \rho}\right)$$

$$\Rightarrow \omega_{12} = \omega_{12}\left(\frac{1}{\rho}\frac{\partial}{\partial \theta}\right)\rho\, d\theta + \omega_{12}\left(\frac{\partial}{\partial \rho}\right) d\rho$$

$$\boxed{\omega_{12} = \frac{1}{\rho}\rho\, d\theta = d\theta}$$

So $\nabla_v e_1 = d\theta(v)\, e_2$

$\nabla_v e_2 = -d\theta(v)\, e_1$

$\theta_1 = 1\, d\rho \Rightarrow d\theta_1 = d1 \wedge d\rho = 0 = \theta_2 \wedge \omega_{21}$

$\theta_2 = \rho\, d\theta \rightarrow d(\theta_2) = d(\rho\, d\theta)$

$= d\rho \wedge d\theta.$

$= \theta_1 \wedge \omega_{12}.$

<u>In general</u>: $d\theta_1 = \omega_{12} \wedge \theta_2$

$d\theta_2 = \omega_{21} \wedge \theta_1$