Solutions to practice problems:

(1) (a) $E_1 = \dfrac{\varphi_u}{\|\varphi_u\|} = (\cos v, \sin v, 0)$.

$$E_2 = \frac{\varphi_v}{\|\varphi_v\|} = \frac{(-u\sin v, u\cos v, b)}{\sqrt{u^2+b^2}}$$

$T_pD \ni (1,0) \longleftrightarrow \varphi_u \qquad (0,1) \longleftrightarrow \varphi_v$.

$du \qquad dv$.

$du(\varphi_u) = 1 \qquad du(\varphi_v) = 0$

$dv(\varphi_u) = 0 \qquad dv(\varphi_v) = 1$

$E_1 = \varphi_u \qquad E_2 = \dfrac{1}{\sqrt{u^2+b^2}}\,\varphi_v$

$\theta_1(E_1) = 1 \qquad \theta_1(E_2) = 0 \implies \theta_1 = du$

$$\left.\begin{array}{l} \theta_2(E_1) = 0 \qquad \theta_2(E_2) = 1 \\ dv(E_2) = \dfrac{1}{\sqrt{u^2+b^2}} \end{array}\right\} \cancel{\implies \theta_2 = dv}$$

$$\implies \theta_2 = \sqrt{u^2+b^2}\, dv$$

$d\theta_1 = \omega_{12} \wedge \theta_2 \implies d\theta_1(E_1, E_2) = \omega_{12}(E_1)$

$d\theta_2 = \omega_{21} \wedge \theta_1 \implies d\theta_2(E_1, E_2) = \omega_{12}(E_2)$

$d\theta_1 = d(du) = 0 \qquad d\theta_2 = d\left(\sqrt{u^2+b^2}\right) \wedge dv$.

$\implies \omega_{12}(E_1) = 0$

$$d\theta_2 = \frac{\partial}{\partial u}\sqrt{u^2+b^2}\; du\wedge dv$$

$$d\theta_2 = \frac{1}{2}\,\frac{2u}{\sqrt{u^2+b^2}}\; du\wedge dv.$$

$$\omega_{12}\,(E_2) = \frac{u}{\sqrt{u^2+b^2}}\; du\wedge dv\,(E_1, E_2)$$

$$= \quad \text{"} \quad du\wedge dv\left(\varphi_u, \frac{\varphi_v}{\sqrt{u^2+b^2}}\right)$$

$$= \frac{u}{u^2+b^2}$$

$$\omega_{12} = \frac{u}{u^2+b^2}\,\theta_2 = \frac{u}{\sqrt{u^2+b^2}}\; dv.$$

$$d\omega_{12} = -K\,\theta_1\wedge\theta_2. = -K\sqrt{u^2+b^2}\; du\wedge dv$$

$$d\omega_{12} = d\left(\frac{u}{\sqrt{u^2+b^2}}\right)\wedge dv.$$

$$= \frac{du}{\sqrt{u^2+b^2}}\wedge dv + u\,d\left(\frac{1}{\sqrt{u^2+b^2}}\right)\wedge dv$$

$$= \frac{du\wedge dv}{\sqrt{u^2+b^2}} + u\left(-\frac{1}{2}\right)\frac{2u}{\left(\sqrt{u^2+b^2}\right)^3}\, du\wedge dv$$

$$\Rightarrow K = \cdots$$

(2) $E = G$, $F = 0$. $\varphi : D \longrightarrow M$.

(a) $\cancel{\forall}\ v, w \in T_pM$.

$\langle v, w\rangle = ?$ or $\|v\| = ?$

$$v = \lambda\varphi_u + \mu\varphi_v \qquad w = \nu\varphi_u + \rho\varphi_v$$

$$\varphi_u\cdot\varphi_u = E = \varphi_v\cdot\varphi_v = G\,,\ \varphi_u\cdot\varphi_v = 0.$$

$$\begin{aligned}\langle v, w\rangle &= \langle \lambda\varphi_u + \mu\varphi_v, \nu\varphi_u + \rho\varphi_v\rangle \\ &= \lambda\nu E + \mu\rho G = E(\lambda\nu + \mu\rho). \\ &= E(v\cdot w)\end{aligned}$$

usual dot product.

So the metric is conformal with scale factor $\lambda = \dfrac{1}{\sqrt{E}} = \dfrac{1}{\sqrt{G}}$.

(b) $E_1 = \dfrac{\varphi_u}{\|\varphi_u\|} = \dfrac{\varphi_u}{\sqrt{E}} = \lambda\varphi_u$

$E_2 = \lambda\varphi_v$. $\theta_1 = \dfrac{1}{\lambda}du$

and $\theta_2 = \dfrac{1}{\lambda}dv$.

$d\theta_1 = \omega_{12}\wedge\cancel{d}\theta_2$.

$$d\theta_1(E_1, E_2) = \omega_{12}(E_1)$$

$$d\theta_1 = d\left(\frac{1}{\lambda}\right)\wedge du = +\frac{1}{\lambda^2}\lambda_v\, du\wedge dv$$

$$\Rightarrow \omega_{12}(E_1) = \lambda_v$$

$$\omega_{21}(E_2) = d\theta_2(E_1, E_2) = \cancel{-}\lambda_u$$

$$\Rightarrow \omega_{12} = \lambda_v\theta_1 \cancel{-} + \lambda_u\theta_2$$

$$= \frac{\lambda_v}{\lambda}du + \frac{\lambda_u}{\lambda}dv.$$

$$d\omega_{12} = d\left(\frac{\lambda_v}{\lambda}\right)\wedge du + d\left(\frac{\lambda_u}{\lambda}\right)\wedge dv.$$

$$= -K\,\theta_1\wedge\theta_2 = -\frac{K}{\lambda^2}du\wedge dv.$$

(c) $H = \frac{1}{2}\left(\omega_{13}(E_1) + \omega_{23}(E_2)\right)$

$E_1 = \lambda \varphi_u \qquad E_2 = \lambda \varphi_v$

$$\nabla_{\varphi_u} E_1 = (E_1)_u = (\lambda\varphi_u)_u = \lambda_u \varphi_u + \lambda \varphi_{uu}$$

$$= \omega_{12}(\varphi_u) E_2 + \omega_{13}(\varphi_u) E_3$$

$$\nabla_{\varphi_v} E_2 = \omega_{21}(\varphi_v) E_1 + \omega_{23}(\varphi_v) E_3$$

$$= (E_2)_v = \cancel{\lambda_v E_2}(\lambda \varphi_v)_v$$

$$= \lambda_v \varphi_v + \lambda \varphi_{vv}$$

$$\nabla_{\varphi_u} E_1 = \frac{1}{\lambda}\omega_{12}(E_1)E_2 + \frac{1}{\lambda}\omega_{13}(E_1)E_3.$$

$$\nabla_{\varphi_v} E_2 = \frac{1}{\lambda}\omega_{21}(E_2)E_1 + \frac{1}{\lambda}\omega_{23}(E_2)E_3.$$

~~add~~ $\cancel{\Rightarrow (E_1)_u + (E_2)_v = \frac{1}{\lambda}\left(\omega_{12}(E_1) \ldots\right.}$

$$\cancel{= \frac{1}{\lambda}\left(\omega_{12}(E_1) - \omega_{12}(E_2)\right)E_2}$$

$$\cancel{+ \frac{1}{\lambda}\left(\omega_{13}(E_1) + \omega_{23}(E_2)\right)E_3}$$

Solve: $\frac{1}{\lambda}\omega_{13}(E_1)E_3 =$

$$= (E_1)_u - \frac{1}{\lambda}\omega_{12}(E_1)E_2.$$

$$= \lambda_u \varphi_u + \lambda \varphi_{uu} - \frac{\lambda_v}{\lambda}E_2$$

$$\frac{1}{\lambda}\omega_{23}(E_2)E_3 = (E_2)_v - \frac{1}{\lambda}\omega_{21}(E_2)E_1$$

$$= \lambda_v \varphi_v + \lambda \varphi_{vv} \mp \frac{\lambda_u}{\lambda}E_1$$

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add: $\frac{1}{\lambda}\left(\omega_{13}(E_1) + \omega_{23}(E_2)\right)E_3 =$

$$= \frac{\lambda_u}{\lambda}E_1 + \lambda\varphi_{uu} - \frac{\lambda_v}{\lambda}E_2$$

$$+ \frac{\lambda_v}{\lambda}E_2 + \lambda\varphi_{vv} \mp \frac{\lambda_u}{\lambda}E_1$$

$$= \lambda\left(\varphi_{uu} + \varphi_{vv}\right)$$

$$0 = H \iff \lambda(\varphi_{uu} + \varphi_{vv}) = 0 \iff \varphi_{uu} + \varphi_{vv} = 0.$$

$$(\lambda \neq 0)$$

(3) $\varphi : D \to M$ principal chart

$S(\varphi_u) = k_1 \varphi_u \quad S(\varphi_v) = k_2 \varphi_v$

$k_1 \neq k_2 \Rightarrow \varphi_u \perp \varphi_v$.

(a) $E_1 = \dfrac{\varphi_u}{\|\varphi_u\|} = \dfrac{\varphi_u}{\sqrt{E}}, \quad E_2 = \dfrac{\varphi_v}{\sqrt{G}}$.

$S(E_1) = k_1 E_1 \quad S(E_2) = k_2 E_2$.

$$\nabla_{\varphi_u} E_3 = \omega_{31}(\varphi_u) E_1 + \omega_{32}(\varphi_u) E_2.$$

$$\overset{\shortparallel}{-S(\varphi_u)} = -k_1 \varphi_u = -k_1 \sqrt{E}\, E_1$$

$\Rightarrow \omega_{13}(\varphi_u) = k_1 \sqrt{E}$

$\omega_{23}(\varphi_u) = 0$

$$\nabla_{\varphi_v} E_3 = \omega_{31}(\varphi_v) E_1 + \omega_{32}(\varphi_v) E_2$$

$$\overset{\shortparallel}{-S(\varphi_v)} = -k_2 \varphi_v = -k_2 \sqrt{G}\, E_2$$

$\Rightarrow \omega_{31}(\varphi_v) = 0$

$\omega_{23}(\varphi_v) = k_2 \sqrt{G}$

$\Rightarrow \omega_{13} = k_1 \sqrt{E}\, du$

$\omega_{23} = k_2 \sqrt{G}\, dv$

$$\ell = S(\varphi_u)\cdot\varphi_u = k_1\,\varphi_u\cdot\varphi_u = k_1 E$$

$$n = S(\varphi_v)\cdot\varphi_v = k_2\,\varphi_v\cdot\varphi_v = k_2 G.$$

$$\Rightarrow \quad \omega_{13} = \frac{\ell}{\sqrt{E}}\,du$$

$$\omega_{23} = \frac{n}{\sqrt{G}}\,dv$$

(b) use part(a) together with the Codazzi equations:

$$d\omega_{13} = \omega_{12}\wedge\omega_{23}$$

$$d\omega_{23} = \omega_{21}\wedge\omega_{13}$$

$$H = \frac{1}{2}\left(\omega_{13}(E_1) + \omega_{23}(E_2)\right)$$

$$\theta_1 = \sqrt{E}\,du \qquad \theta_2 = \sqrt{G}\,dv.$$

$$\Rightarrow \omega_{12} = \frac{1}{2\sqrt{EG}}\left(E_v\,du + G_u\,dv\right)$$

(compute like before)

$$d\omega_{13} = d\left(\frac{\ell}{\sqrt{E}}\right)\wedge du$$

$$= \left(-\frac{\ell_v}{\sqrt{E}} + \frac{1}{2}\frac{E_v}{\sqrt{E^3}}\ell\right)du\wedge dv.$$

$$d\omega_{23} = d\left(\frac{n}{\sqrt{G}}\right) dv$$

$$= \left(-\frac{n_u}{\sqrt{G}} + \frac{1}{2}\frac{G_u}{\sqrt{G^3}}\, n\right) du \wedge dv$$

$$d\omega_{13} = \omega_{12} \wedge \omega_{23}.$$

$$-\frac{l_v}{\sqrt{E}} + \frac{1}{2}\frac{l E_v}{\sqrt{E^3}} = -\frac{n}{2\sqrt{EG}} E_v$$

$$-\frac{n_u}{\sqrt{G}} + \frac{1}{2}\frac{n G_u}{\sqrt{G^3}} = -\frac{l}{2\sqrt{GE}} G_u$$

Solve for l_v, n_u:

$$l_v = \frac{1}{2}\left(\frac{n}{G} + \frac{l}{E}\right) E_v$$

$$n_u = \frac{1}{2}\left(\frac{l}{E} + \frac{n}{G}\right) G_u$$

$$l_v = \frac{1}{2}(k_1 + k_2) E_v$$

$$n_u = \frac{1}{2}(k_1 + k_2) G_u$$

$$\frac{1}{2}(k_1 + k_2) = H$$

(4) P = Poincaré half plane:

$D = M = \{(x,y) : y > 0\}$.

<u>metric</u>: $\langle v, w \rangle = \frac{1}{y^2} v \cdot w$.

(a) $\alpha(t) = (r\cos t, r\sin t)$

$0 < t < \pi$

speed $\|\alpha'(t)\| = \sqrt{\frac{\alpha'(t)\cdot\alpha'(t)}{y^2}}$

$= \sqrt{\frac{\alpha'(t)\cdot\alpha'(t)}{r^2\sin^2 t}} = \frac{1}{\sin t}$.

(b) length: $\int_0^{\pi} \frac{dt}{\sin t}$.

$= \lim_{s\to 0} \int_s^{\pi/2} \frac{dt}{\sin t} + \lim_{s\to\pi} \int_{\pi/2}^{s} \frac{dt}{\sin t}$

$\frac{dt}{\sin t} = \frac{\sin t\, dt}{\sin^2 t} = \frac{-d(\cos t)}{1-\cos^2 t}$

$v = \cos t$.

(c) <u>area</u>: $\displaystyle\iint_{\text{half circle}} \sqrt{EG - F^2}\, dx\, dy$

$$= \iint_{\text{half circle}} \frac{1}{y}\, dx\, dy.$$