we prove $\omega = \omega_{12}$ :

By our observation, we need to show: $\omega(E_1) = \omega_{12}(E_1)$ and

$\omega(E_2) = \omega_{12}(E_2)$.

$$d\theta_1(E_1, E_2) = (\omega \wedge \theta_2)(E_1, E_2)$$
$$= \omega(E_1)\,\theta_2(E_2) = \omega(E_1)$$
$$= (\omega_{12} \wedge \theta_2)(E_1, E_2) = \omega_{12}(E_1)$$

$$d\theta_2(E_1, E_2) = -(\omega \wedge \theta_1)(E_1, E_2)$$
$$= +\omega(E_2)\,\theta_1(E_1) = \omega(E_2)$$
$$= -(\omega_{12} \wedge \theta_2)(E_1, E_2) = \omega_{12}(E_2)$$

So $\omega = \omega_{12}$ $\square$.

---

<u>Back to an abstract surface M :</u>

$\varphi : D \longrightarrow M$ a coordinate chart.

$p \in M$ $\qquad q = \varphi^{-1}(p)$.

$$\mathbb{R}^2 = T_q D \xrightarrow{\cong} T_p M$$
$$(1,0) \longleftrightarrow \varphi_u$$
$$(0,1) \longleftrightarrow \varphi_v$$

Suppose we are given a metric $\langle\, , \rangle$ on $M$

**Can** Define $E_1 := \dfrac{1}{\|\varphi_u\|}\varphi_u$

$$E_2 := \frac{1}{\left\|\varphi_v - \frac{\langle \varphi_u, \varphi_v\rangle}{\langle \varphi_u, \varphi_u\rangle}\varphi_u\right\|}\left(\varphi_v - \frac{\langle \varphi_u, \varphi_v\rangle}{\langle \varphi_u, \varphi_u\rangle}\varphi_u\right)$$

So frame fields exist on $M$ (there are lots of them).

Given any frame field $\{E_1, E_2\}$ on $M$, we can define the dual 1-forms $\theta_1, \theta_2$.

$\forall\ p \in M,\quad V \in T_pM$

can write $V = \lambda E_1 + \mu E_2$.

so $\theta_1(V) = \lambda\ ,\quad \theta_2(V) = \mu$

Can define $\omega_{12}$ by the relations:

$$\boxed{d\theta_1 = \omega_{12}\wedge\theta_2\ ,\quad d\theta_2 = \omega_{21}\wedge\theta_1}$$

$$\Rightarrow\quad d\theta_1(E_1,E_2) = (\omega_{12}\wedge\theta_2)(E_1,E_2) = \omega_{12}(E_1)$$

$$d\theta_2(E_1,E_2) = (\omega_{21}\wedge\theta_1)(E_1,E_2) = -\omega_{21}(E_2) = \omega_{12}(E_2)$$

$$\boxed{\omega_{12} = \omega_{12}(E_1)\theta_1 + \omega_{12}(E_2)\theta_2.}$$

<u>Recall</u>: $d\omega_{12} = -K\,\theta_1 \wedge \theta_2$.

$\theta_1 \wedge \theta_2$ gives a basis of the space of two forms, so $d\omega_{12}$ is a multiple of $\theta_1 \wedge \theta_2$, the Gaussian curvature is defined to be the negative of the coefficient of $d\omega_{12}$ on $\theta_1 \wedge \theta_2$.

---

When $M$ was in $\mathbb{R}^3$, we had a unit normal, say $E_3$. For any two vector fields $V, W$ on $M$, we could extend them to a small neighborhood of $M$ and then define the <u>covariant derivative</u> of $W$ in the direction of $V$ to be the projection to

$T_pM$ (at any $p \in M$) of the directional derivative of $W$ in the direction of $V$.

$\tilde{\nabla}_V W :=$ the directional derivative of $W$ in $\mathbb{R}^3$.

$\{E_1, E_2, E_3\}$ frame field on $\mathbb{R}^3$ adapted to $M$.

$\tilde{\nabla}_V W = ?\,E_1 + ?\,E_2 + ?\,E_3.$

$\nabla_V W := ?\,E_1 + ?\,E_2$

---

Need to generalize the covariant derivative to an abstract geometric surface. We need certain properties of the derivative, (so it behaves like what we intuitively think of as a derivative)

**Def:** A covariant derivative on $M$ is a rule for assigning a vector field $\nabla_Y W$ to a pair of vector fields $Y, W$ on $M$. This should have the following properties:

(1) It should be linear in $Y$:

$\forall$ functions $f_1, f_2$ on $M$

$\forall$ vector fields $V_1, V_2$ on $M$:

$\forall$ $W$ vector field on $M$:

$$\nabla_{(f_1 V_1 + f_2 V_2)} W = f_1 \nabla_{V_1} W + f_2 \nabla_{V_2} W.$$

(2) $\forall$ $V$ vector field on $M$

$\forall$ ~~functions~~ $W_1, W_2$ vector fields on $M$:

$$\nabla_V (W_1 + W_2) = \nabla_V (W_1) + \nabla_V (W_2).$$

(3) $\forall$ $V$ vector field on $M$

$\forall$ $f$ function on $M$

$\forall$ $W$ vector field on $M$

$$\nabla_V (fW) = (\nabla_V (f)) W + f(\nabla_V W)$$

in O'Neill: $V[f] := \nabla_V (f)$

given a curve $\alpha(t)$ s.t.

$\alpha(0) = p$. $\quad \alpha'(0) = V$

$$\nabla_V (f) \text{ at } p \quad = \frac{d}{dt} f(\alpha(t))\Big|_{t=0}$$

(p. 155 in O'Neill, Def. 4.3.10)

(4) $\forall$ $V$ vector field on $M$

$\forall$ $W_1, W_2$ vector fields on $M$

$$\nabla_V (\langle W_1, W_2 \rangle) = V[\langle W_1, W_2 \rangle]$$

$$= \langle \nabla_V W_1, W_2 \rangle + \langle W_1, \nabla_V W_2 \rangle$$

---

<u>Remark</u>: To define a covariant derivative on $M$, using the above

properties, it is enough to define it on $E_1, E_2$ where $\{E_1, E_2\}$ is a frame field on $M$:

Suppose we know $\nabla_V(E_1)$ and $\nabla_V(E_2)$

given any vector field $W$ on $M$, we can find functions $f_1, f_2$ on $M$ s.t. $W = f_1 E_1 + f_2 E_2$

then $\nabla_V W = \nabla_V (f_1 E_1) + \nabla_V (f_2 E_2)$

$= V[f_1] E_1 + f_1 \nabla_V (E_1) + V[f_2] E_2 + f_2 \nabla_V (E_2)$

We already have $E_1, E_2, \theta_1, \theta_2$, $\omega_{12}$, so we can define

$\nabla_V E_1 := \omega_{12}(V) E_2$

$\nabla_V E_2 := \omega_{21}(V) E_1 = -\omega_{12}(V) E_1$

---

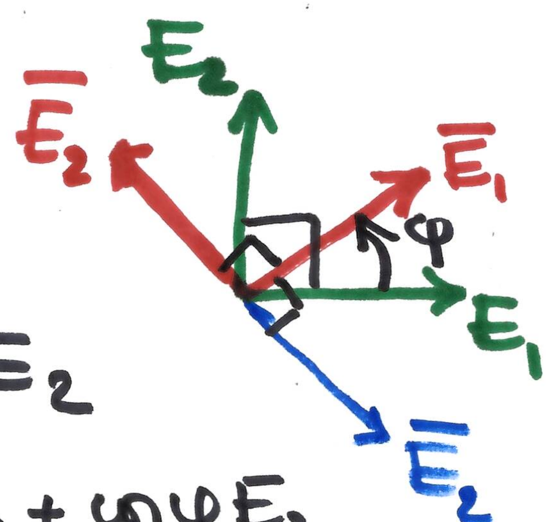
given a curve in $M$, say $\alpha(t)$, extend $\alpha'(t)$ to a vector field $V$

in a neighborhood of $\alpha$.

<u>Def</u>: We say that a vector field $W$ on $M$, is parallel on $\alpha$ if $\nabla_V W = 0$ everywhere on $\alpha$. (In other words, when we restrict $W$ to $\alpha$, $W' = 0$)

---

What happens when we change frame fields?




$\cos\varphi := \langle E_1, \bar{E}_1 \rangle$

$\bar{E}_1 = \cos\varphi\, E_1 + \sin\varphi\, E_2$

<u>either</u>: $\bar{E}_2 = -\sin\varphi\, E_1 + \cos\varphi\, E_2$

<u>or</u>: $\bar{E}_2 = \sin\varphi\, E_1 - \cos\varphi\, E_2$.

$\varphi(t)$ is a function on $M$.

We'll do the case where $\{\bar{E}_1, \bar{E}_2\}$ has the same orientation as $\{E_1, E_2\}$, the other case is similar.

$$\bar\theta_1 = \bar\theta_1(E_1)\,\theta_1 + \bar\theta_1(E_2)\,\theta_2$$

$$\begin{pmatrix} \bar E_1 \\ \bar E_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}^{-1} \begin{pmatrix} \bar E_1 \\ \bar E_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \bar E_1 \\ \bar E_2 \end{pmatrix}$$

$$\Rightarrow \bar\theta_1 = \cos\varphi\,\theta_1 + \sin\varphi\,\theta_2$$

$$\bar\theta_2 = \bar\theta_2(E_1)\,\theta_1 + \bar\theta_2(E_2)\,\theta_2$$

$$= -\sin\varphi\,\theta_1 + \cos\varphi\,\theta_2.$$

$$\begin{pmatrix} \bar\theta_1 \\ \bar\theta_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$$

$$d\bar\theta_1 = d(\cos\varphi\,\theta_1) + d(\sin\varphi\,\theta_2)$$

$$= -\sin\varphi\, d\varphi \wedge \theta_1 + \cos\varphi\, d\theta_1$$

$$+\cos\varphi \, d\varphi \wedge \theta_2 + \sin\varphi \, d\theta_2 .$$

$$d\theta_2 = -\cos\varphi \, d\varphi \wedge \theta_1 - \sin\varphi \, d\theta_1$$

$$- \sin\varphi \, d\varphi \wedge \theta_2 + \cos\varphi \, d\theta_2 .$$