$$+\cos\varphi\, d\varphi\wedge\theta_2 + \sin\varphi\, d\theta_2.$$

$$d\bar\theta_2 = -\cos\varphi\, d\varphi\wedge\theta_1 - \sin\varphi\, d\theta_1$$
$$-\sin\varphi\, d\varphi\wedge\theta_2 + \cos\varphi\, d\theta_2.$$

$$d\theta_1 = \omega_{12}\wedge\theta_2 \ , \quad d\theta_2 = \omega_{21}\wedge\theta_1$$
$$d\bar\theta_1 = \bar\omega_{12}\wedge\bar\theta_2 \ , \quad d\bar\theta_2 = \bar\omega_{21}\wedge\bar\theta_1$$

$$\Rightarrow\ d\bar\theta_1 = -\sin\varphi\, d\varphi\wedge\theta_1 + \cos\varphi\, d\varphi\wedge\theta_2$$
$$+\cos\varphi\,\omega_{12}\wedge\theta_2 + \sin\varphi\,\omega_{21}\wedge\theta_1$$
$$= \sin\varphi(\omega_{21} - d\varphi)\wedge\theta_1$$
$$+\cos\varphi(\omega_{12}+d\varphi)\wedge\theta_2$$
$$= -\sin\varphi(\omega_{12}+d\varphi)\wedge\theta_1$$
$$+\cos\varphi(\omega_{12}+d\varphi)\wedge\theta_2$$
$$= (\omega_{12}+d\varphi)\wedge(-\sin\varphi\,\theta_1 + \cos\varphi\,\theta_2)$$

$$d\bar\theta_1 = (\omega_{12}+d\varphi)\wedge\bar\theta_2$$

Similarly: $d\bar\theta_2 = -(\omega_{12}+d\varphi)\wedge\bar\theta_1$

So: $\boxed{\bar\omega_{12} = \omega_{12}+d\varphi}$ ($= -\omega_{12} - d\varphi$ if opposite orientation)

Also $\bar\theta_1\wedge\bar\theta_2 = \det\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}\theta_1\wedge\theta_2$

$$\boxed{\bar\theta_1 \wedge \bar\theta_2 = \theta_1 \wedge \theta_2} \quad \left(= -\theta_1 \wedge \theta_2 \text{ if opposite orientation}\right)$$

<u>First consequence</u>: The Gaussian curvature is well-defined.

In other words, ~~for any~~ if

$$d\omega_{12} = -K\,\theta_1 \wedge \theta_2$$

for one frame field, we also have $d\bar\omega_{12} = -K\,\bar\theta_1 \wedge \bar\theta_2$ <u>for</u> any other frame field.

<u>Proof</u>: We already know:

$$\bar\theta_1 \wedge \bar\theta_2 = \pm\theta_1 \wedge \theta_2$$

also $\bar\omega_{12} = \pm(\omega_{12} + d\varphi)$

so
$$\begin{aligned} d\bar\omega_{12} &= \pm\, d(\omega_{12} + d\varphi) \\ &= \pm(d\omega_{12} + d(d\varphi)) \\ &= \pm\, d\omega_{12} \end{aligned}$$

Second consequence: The covariant derivative is well-defined.
Meaning, it does not depend on the choice of frame field.

Proof: recall: $\nabla_V(E_1) = \omega_{12}(V)E_2$

$$\nabla_V(E_2) = \omega_{21}(V)E_1$$

for the second frame field, we have

$$\overline{\nabla}_V(\overline{E}_1) = \overline{\omega}_{12}(V)\,\overline{E}_2$$

$$\overline{\nabla}_V(\overline{E}_2) = \overline{\omega}_{21}(V)\,\overline{E}_1$$

Because ∇ & $\overline{\nabla}$ both satisfy the Leibnitz rules, to show they are equal on all vector fields W, it is enough to show:

$$\overline{\nabla}_V(E_1) = \nabla_V(E_1) \qquad \forall V$$

and $\overline{\nabla}_V(E_2) = \nabla_V(E_2)$

$$\overline{\nabla}_V(E_1) = \overline{\nabla}_V(\cos\varphi\,\overline{E}_1 - \sin\varphi\,\overline{E}_2)$$

$$= \overline{\nabla}_V(\cos\varphi)\,\overline{E}_1 + \cos\varphi\,\overline{\nabla}_V(\overline{E}_1)$$

$$- \overline{\nabla}_V(\sin\varphi)\,\overline{E}_2 - \sin\varphi\,\overline{\nabla}_V(\overline{E}_2)$$

$$= -\sin\varphi\,\varphi[V]\,\overline{E}_1 + \cos\varphi\,\overline{\omega}_{12}(V)\,\overline{E}_2$$

$$- \cos\varphi\,\varphi[V]\,\overline{E}_2 - \sin\varphi\,\overline{\omega}_{21}(V)\,\overline{E}_1$$

<u>claim:</u> $\varphi[V] = d\varphi(V)$

in a coordinate chart: $\varphi(u,v)$

$$d\varphi = \frac{\partial\varphi}{\partial u}\,du + \frac{\partial\varphi}{\partial v}\,dv.$$

$$V = a\,\varphi_u + b\,\varphi_v \text{ "="} a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}\text{"}$$

$$\varphi[V] = a\frac{\partial\varphi}{\partial u} + b\frac{\partial\varphi}{\partial v}$$

$$= d\varphi(V)$$

So $\overline{\nabla}_V(E_1) = -\sin\varphi\,d\varphi(V)\,\overline{E}_1$

$+ \cos\varphi\,((\omega_{12}+d\varphi)(V))\,\overline{E}_2 - \cos\varphi\,(d\varphi(V))\,\overline{E}_2$

$+ \sin\varphi\,(\omega_{12}+d\varphi)(V)\,\overline{E}_1$

$$\overline{\nabla}_V(E_1) = \cos\varphi\ \omega_{12}(V)\ \overline{E}_2$$
$$+ \sin\varphi\ \omega_{12}(V)\ \overline{E}_1$$

$$= \omega_{12}(V)\left(\cos\varphi\ \overline{E}_2 + \sin\varphi\ \overline{E}_1\right)$$

$$= \omega_{12}(V)\ E_2 = \nabla_V(E_1)$$

Similarly $\overline{\nabla}_V(E_2) = \nabla_V(E_2)$.

α a curve on M.
First property of a vector field W parallel on α: W has constant length along α:

$$\|W\|^2 = \langle W, W\rangle$$

and
$$\left(\|W\|^2\right)' = \langle W, W\rangle'$$
$$= \nabla_V \langle W, W\rangle$$
$$= \langle \nabla_V W, W\rangle + \langle W, \nabla_V W\rangle$$
$$= 2\langle \nabla_V W, W\rangle = 0$$
if W is parallel.

Rotation operator on M:

Choose an orientation on M.

This could be the orientation of a frame field,
or it could be an area form:

$$\theta_1 \wedge \theta_2 \longleftrightarrow E_1 \times E_2$$

an area form is a 2-form Ω s.t.

$$\iint_{\text{subset of } M} \Omega = \text{area (subset of M)}$$

in coordinates: ~~Ω~~ u, v.

$$\Omega = f(u,v)\, du \wedge dv.$$

$$\iint_? \Omega = \iint_? f(u,v)\, du\, dv$$

last quarter: $\Omega = \sqrt{EG - F^2}\, du\, dv$

$\forall \{E_1, E_2\}$ frame field with given orientation

$$J(E_1) = E_2 \qquad J(E_2) = -E_1$$

rotation by $\pi/2$ in $T_p M$.

linear operator : $T_pM \longrightarrow T_pM$.

$$J^2 = -Id$$

"Almost complex structure"

<u>Lemma:</u> Suppose Y is a vector field of constant length on a curve α in M. Then:

$$Y' = (\varphi' + \omega_{12}(\alpha'))\, J(Y)$$

where φ is the angle of Y with E_1, E_1 ~~of to~~ a vector from a frame field $\{E_1, E_2\}$.

<u>Proposition:</u> Given α and $t_0 \in \mathbb{R}$ where α is defined with $\alpha(t_0) = p \in M$, given a vector $V_0 \in T_pM$, $\exists!$ vector field V parallel on α with $V(t_0) = V_0$.

Proof of the lemma:

put $\quad c =$ length of Y

then $\quad Y = c(\cos\varphi\, E_1 + \sin\varphi\, E_2)$

$$Y' = c\Big(-\sin\varphi\,\varphi'\,E_1 + \cos\varphi\,\omega_{12}(\alpha')E_2 + \cos\varphi\,\varphi'\,E_2 + \sin\varphi\,\omega_{21}(\alpha')E_1\Big)$$

$$= c\Big[\big(\varphi' + \omega_{12}(\alpha')\big)\big(-\sin\varphi\,E_1\big) + \big(\varphi' + \omega_{12}(\alpha')\big)\big(\cos\varphi\,E_2\big)\Big]$$

$$= c\,\big(\varphi' + \omega_{12}(\alpha')\big)\big(\cos\varphi\,E_2 - \sin\varphi\,E_1\big)$$

$$= \big(\varphi' + \omega_{12}(\alpha')\big)\,J(Y) \qquad \square.$$

Proof of the proposition:

By the lemma, a vector field

$$V = c\,(\cos\varphi\,E_1 + \sin\varphi\,E_2)$$

of constant length is parallel

$\iff \quad \varphi' + \omega_{12}(\alpha') = 0$

note: $c = \|V_0\|$ is given.

So to obtain V, we need φ.

the equation gives $\varphi' = -\omega_{12}(\alpha')$

So $$\varphi = \int_{t_0}^{t} -\omega_{12}(\alpha')$$

$$\varphi(t_0) = \text{angle of } V_0 \text{ with } E_1$$

$$\Rightarrow \cos\varphi(t_0) = \frac{\langle E_1, V_0 \rangle}{\|V_0\|} \qquad \square$$