Practice problems:

(2) The Poincaré plane:

upper half plane $\{(u,v) \mid v>0\}$

with metric: $\langle w_1, w_2\rangle := \dfrac{w_1\cdot w_2}{v^2}$

at a point $p=(u,v)\in$ half plane

$E_1 = v\,U_1 \qquad E_2 = v\,U_2$

(a) $\theta_1 = \frac{1}{v}\,du \qquad \theta_2 = \frac{1}{v}\,dv$.

$$\omega_{12} = \omega_{12}(E_1)\theta_1 + \omega_{12}(E_2)\theta_2$$

$$\omega_{12}(E_1) = d\theta_1(E_1, E_2)$$

$$\omega_{12}(E_2) = d\theta_2(E_1, E_2)$$

$$d\theta_1 = d\left(\frac{1}{v}\,du\right) = d\left(\frac{1}{v}\right)\wedge du$$

$$= \left(\frac{\partial}{\partial u}\left(\frac{1}{v}\right)du + \frac{\partial}{\partial v}\left(\frac{1}{v}\right)dv\right)\wedge du$$

$$= \frac{1}{v^2}\,du\wedge dv = \theta_1\wedge\theta_2$$

$$d\theta_2 = d\left(\frac{1}{v}\,dv\right) = d\left(\frac{1}{v}\right)\wedge dv$$

$$= 0$$

$$\omega_{12} = \theta_1 = \frac{1}{v}\,du$$

(2) Covariant derivative:

$$\nabla_V E_1 = \omega_{12}(V)\, E_2$$

$$\nabla_V E_2 = -\,\omega_{12}(V)\, E_1$$

$$\alpha(t) = r(\cos t, \sin t) \qquad 0<t<\pi.$$

$$\alpha'(t) = r(-\sin t, \cos t).$$

$$= r\left(-\sin t\, U_1 + \cos t\, U_2\right)$$

$$= \frac{r}{v}\left(-\sin t\, E_1 + \cos t\, E_2\right).$$

$$\alpha''(t) = \nabla_{\alpha'(t)}\left(\alpha'(t)\right)$$

$$\alpha'(t) = \frac{r}{r\sin t}\left(-\sin t\, E_1 + \cos t\, E_2\right)$$

$$= -E_1 + \cot t\, E_2$$

$$\nabla_{\alpha'}(\alpha') = -\nabla_{\alpha'} E_1 + \frac{d}{dt}(\cot t)\, E_2$$

$$+ \cot t\, \nabla_{\alpha'} E_2$$

$$\nabla_{\alpha'} E_1 = \omega_{12}(\alpha'(t))\, E_2.$$

$$= \frac{1}{v}\, du(\alpha'(t))\, E_2.$$

$$= \frac{1}{r\sin t}\cdot(-r\sin t)\, E_2 = -E_2$$

$$\nabla_{\alpha'} E_2 = -\omega_{12}(\alpha'(t))\, E_1$$
$$= \frac{-1}{v}\, du\ (\alpha'(t))\, E_1$$
$$= E_1$$

$$\alpha''(t) = E_2 - \frac{1}{\sin^2 t} E_2 + \cot t\ E_1$$
$$= -\cot^2 t\ E_2 + \cot t\ E_1$$

(c) $\beta(t) = (c, s)\, t = t(c, s)$

$c^2 + s^2 = 1 \qquad st > 0$

$$\alpha'(t) = (c, s) = c U_1 + s U_2$$

~~acceleration:~~ $\cancel{\alpha''(t) =}$

$$= \frac{c}{v} E_1 + \frac{s}{v} E_2$$
$$= \frac{c}{ts} E_1 + \frac{s}{ts} E_2$$
$$= \frac{c}{s}\frac{1}{t} E_1 + \frac{1}{t} E_2.$$
$$= \frac{1}{t}\left(\frac{c}{s} E_1 + E_2\right)$$

$$\alpha''(t) = \nabla_{\alpha'}(\alpha') = -\frac{1}{t^2}\left(\frac{c}{s} E_1 + E_2\right)$$
$$+ \frac{1}{t}\nabla_{\alpha'}\left(\frac{c}{s} E_1 + E_2\right)$$

$$\alpha''(t) = -\frac{1}{t^2}\left(\frac{c}{s}E_1 + E_2\right) + \frac{1}{t}\left(\frac{c}{s}\frac{1}{v}du(\alpha')E_2 - \frac{1}{v}du(\alpha')E_1\right)$$

$$= -\frac{1}{t^2}\left(\frac{c}{s}E_1 + E_2\right) + \frac{1}{t}\left(\frac{c^2}{ts^2}E_2 - \frac{c}{ts}E_1\right)$$

$$= -\frac{2c}{t^2 s}E_1 + \frac{1}{t^2}\left(\frac{c^2}{s^2} - 1\right)E_2.$$

(3) $\omega_{12} = \theta_1 = \frac{1}{v}du \qquad \theta_2 = \frac{1}{v}dv$

$d\omega_{12} = d\theta_1 = \theta_1 \wedge \theta_2$

$\Rightarrow \quad K = -1$

(4) $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$(u, v) \longmapsto (f(u)\cos v, f(u)\sin v, v)$

image is a helicoid.

(a) pull-back metric: at $(u_0, v_0) = p$

$w = (a, b) = a(1,0) + b(0,1)$

$F_* w = ?$

$F_*(1,0) = \varphi_u$ = tangent vector to u-parameter curve

$= (f'(u)\cos v, f'(u)\sin v, 0)$

$F_*(0,1) = \varphi_v$

$= (f(u)(-\sin v), f(u)\cos v, 1)$

$F_* w = a\varphi_u + b\varphi_v$

metric: $\langle w_1, w_2 \rangle := \langle F_* w_1, F_* w_2 \rangle$

$= (F_* w_1) \cdot (F_* w_2)$

$F_* w_i = a_i \varphi_u + b_i \varphi_v \quad , w_i = (a_i, b_i)$

$\langle w_1, w_2 \rangle = (a_1\varphi_u + b_1\varphi_v) \cdot (a_2\varphi_u + b_2\varphi_v)$

$= a_1 a_2 E + (a_1 b_2 + a_2 b_1) F + b_1 b_2 G$

(b) $E_1 := \dfrac{\varphi_u}{\|\varphi_u\|}$ assume $f' > 0$

$$E_2 := \frac{1}{\left\| \varphi_v - \dfrac{\varphi_u \cdot \varphi_v}{\varphi_u \cdot \varphi_u} \varphi_u \right\|} \left(\varphi_v - \frac{\varphi_u \cdot \varphi_v}{\varphi_u \cdot \varphi_u} \varphi_u \right)$$

$$E_1 = (\cos v, \sin v, 0)$$

$$E = \varphi_u \cdot \varphi_u = f'^2 \qquad F = \varphi_u \cdot \varphi_v = 0$$

$$G = \varphi_v \cdot \varphi_v = f^2 + 1$$

$$E_2 = \frac{1}{\|\varphi_v\|} \varphi_v = \frac{1}{\sqrt{f^2+1}} (-f(u)\sin v, f(u)\cos v, 1)$$

$$\theta_1 = \|\varphi_u\| \, du = f'(u) \, du$$

$$\theta_2 = \|\varphi_v\| \, dv = \sqrt{f^2+1} \; dv$$

$$d\theta_1 = d(f'(u)) \wedge du = 0$$

$$d\theta_2 = d\left(\sqrt{f^2+1}\right) \wedge dv.$$

$$= \frac{f f'}{\sqrt{f^2+1}} \, du \wedge dv.$$

$$= \frac{f}{f^2+1} \, \theta_1 \wedge \theta_2$$

$$\Rightarrow \quad \omega_{12} = \frac{f}{f^2+1} \theta_2 = \frac{f}{\sqrt{f^2+1}} dv$$

$$d\omega_{12} = d\left(\frac{f}{\sqrt{f^2+1}}\right) \wedge dv.$$

$$d\omega_{12} = \frac{f'\sqrt{f^2+1} - f\dfrac{ff'}{\sqrt{f^2+1}}}{f^2+1}\, du \wedge dv$$

$$d\omega_{12} = \frac{f'}{\left(\sqrt{f^2+1}\right)^3}\, du \wedge dv$$

$$\theta_1 \wedge \theta_2 = f'(u)\sqrt{f^2+1}\ du \wedge dv$$

$$\Rightarrow \quad K = -\frac{1}{(f^2+1)^2}$$

Back to the Poincaré half plane:

We are going to find the geodesics

$$E = G = \frac{1}{v^2} \qquad F = 0$$

v- Clairault

geodesic equations:

$$u'' - \frac{2}{v}u'v' = 0 \qquad v'' + \frac{1}{v}u'^2 - \frac{1}{v}v'^2 = 0$$

(1) $u' = 0$ $\qquad u = c$ constant.

vertical line. v-parameter curve.

v-parameter curves are geodesics: this is a v-Clairault patch.

(2) $u' \neq 0$ $\quad \dfrac{u''}{u'} = 2\dfrac{v'}{v}$

integrate $\quad \ln u' = 2 \ln v + \text{const.}$

$$\Rightarrow \quad u' = c v^2 \quad c \text{ constant.}$$

unit speed equation:

$$\frac{u'^2 + v'^2}{v^2} = 1$$

$$\Rightarrow \quad c^2 v^4 + v'^2 = v^2$$

$$v'^2 = v^2 - c^2 v^4 = v^2(1 - c^2 v^2)$$

$$\Rightarrow v' = \pm v\sqrt{1 - c^2 v^2}$$

divide by u' : $\quad \dfrac{v'}{u'} = \dfrac{v}{u'}\sqrt{1 - c^2 v^2}$

$$= \frac{v}{c v^2}\sqrt{1 - c^2 v^2}$$

$$\frac{c v v'}{\sqrt{1 - c^2 v^2}} = u'$$

<u>integrate:</u> $$cu = -\sqrt{1-c^2v^2} + K,$$

$$(cu-K)^2 = 1 - c^2v^2$$

$$(cu-K)^2 + c^2v^2 = 1$$

half circle with center on x-axis.