(1) (26 points) Let $M$ and $N$ be two surfaces and $F : M \to N$ a differentiable map. Let $p$ be a point of $M$ and $T_p M$ the tangent plane to $M$ at $p$.

(a) Give the definition of the differential of $F$ at $p$.

(b) Give the definition of the exponential map $\exp_p : T_p M \to M$.

(c) Give the definition of a normal neighborhood of $p$.

(d) Give the definition of a geodesic polar parametrization.

Solution:

(a) Let $v \in T_p M$ be a tangent vector.

Let $\alpha(t) = \varphi(u(t), v(t))$ be a curve in $M$ such that, $\alpha(0) = p$, $\alpha'(0) = v$. The composition $\beta(t) = F(\alpha(t))$ is then a curve in $N$ and we define

$$F_*(v) := \beta'(0) = \frac{d}{dt} F(\alpha(t))|_{t=0}.$$ 

(b) Let $v$ be a tangent vector to $M$ at $p$ and let $\gamma_v$ be the unique geodesic with $\gamma_v(0) = p, \gamma_v'(0) = v$. Choose $v$ small enough so that $\gamma_v$ is defined on the interval $[-1, 1]$. Then we define

$$\exp_p(v) := \gamma_v(1).$$ 

(c) A normal neighborhood is the image $\mathcal{N}_\varepsilon$ of the disc $D_\varepsilon$ of radius $\varepsilon$ and center $0$ by the exponential map, provided that the exponential map is a diffeomorphism from $D_\varepsilon$ to $\mathcal{N}_\varepsilon$.

(d) Choose a basis $\{e_1, e_2\}$ of $T_p M$. The map

$$\varphi(u, v) := \exp_p(u \cos ve_1 + u \sin ve_2) = \gamma_{u \cos ve_1 + u \sin ve_2}(1) = \gamma_{u \cos ve_1 + u \sin ve_2}(u)$$

is called a geodesic polar map.

(2) (24 points) Prove that for a geodesic polar parametrization $\varphi(u, v)$ of $M$ with pole $p$, we have $E = 1, F = 0$ everywhere and $G > 0$ everywhere except at $p$.

Solution: From the definition of the exponential map, we have $E = \varphi'_u \varphi_u = \gamma'_{\cos ve_1 + \sin ve_2}(u)$.

$\gamma'_{\cos ve_1 + \sin ve_2}(u) = 1$ because $\gamma_{\cos ve_1 + \sin ve_2}(u)$ is a unit speed geodesic. Furthermore, since
\( \varphi(u, v_0) = \gamma_{\cos v_0 e_1 + \sin v_0 e_2}(u) \) is a unit speed geodesic, \( \varphi_{uu} \) (the acceleration vector) is normal to \( M \) everywhere. In particular, \( E_v = \varphi_u \cdot \varphi_{uv} = 0 \) and \( \varphi_{uu} \cdot \varphi_v = \varphi_{uu} \cdot \varphi_u = 0 \). Next

\[ F_u = (\varphi_u \cdot \varphi_v)_u = \varphi_{uu} \cdot \varphi_v + \varphi_u \cdot \varphi_{uv} = 0. \]

So \( F \) is constant on each \( u \)-parameter curve. Since \( \varphi(0, v) = p \) for all \( v \), we have \( \varphi_v(0, v) = 0 \) for all \( v \), hence \( F(0, v) = 0 \) and \( F \) is zero everywhere. Finally, since we have a coordinate chart away from \( p \), the length \( |\varphi_v| = \sqrt{G} > 0 \), so \( G > 0 \).

3. (25 points)

(a) For the cone \( z^2 = 4x^2 + y^2 \), write a ruling parametrization of the form \( \beta(u) + v\delta(u) \).

(b) Do the same for the cylinder \( x^2 + y^2 = 4 \).

(c) Find all the geodesics on the cylinder \( x^2 + y^2 = 4 \) in parametric form.

**Solution:**

(a) Every line in the cone goes through the origin. If we cut the cone with the circle at \( z = 1 \), we obtain vectors parallel to each line: \( ((\cos u)/2, \sin u, 1) \). So a parametrization would be

\[ \varphi(u, v) = v((\cos u)/2, \sin u, 1). \]

(b) For the cylinder, all lines are parallel to \( (0, 0, 1) \) and they go through points on the circle \( x^2 + y^2 = 4, z = 0 \). So a parametrization would be

\[ \varphi(u, v) = 2(\cos u, \sin u, 0) + v(0, 0, 1). \]

(c) First note that in the parametrization above \( \varphi_u = (-2\sin u, 2\cos u, 0), \varphi_v = (0, 0, 1) \) So \( E = 4, F = 0, G = 1 \). Let \( \alpha(t) = \varphi(u(t), v(t)) = (2\cos u(t), 2\sin u(t), v(t)) \) be a unit speed geodesic on the cylinder. Then the geodesic equations give us

\[ u'' = v'' = 0 \]

So \( u(t) = at + b, v(t) = ct + d \) for some constants \( a, b, c, d \).

4. (25 points) Recall that a ruled surface \( M \) is developable if it has a parametrization \( \varphi(u, v) = \beta(u) + v\delta(u) \) such that the unit normal \( U \) is independent of \( v \). Recall that \( M \) is developable if and only if its Gaussian curvature is 0.

(a) Write two different ruling parametrizations for the saddle surface \( z = xy \) to show that it is doubly ruled.

(b) Is the saddle surface developable?
Solution:

(a) A parametrization of the surface is $\varphi(u, v) = (u, v, uv)$. So one ruling is

$$
\varphi(u, v) = (u, 0, 0) + v(0, 1, u).
$$

Another parametrization is $\varphi(u, v) = (v, u, uv)$. So another ruling is

$$
\varphi(u, v) = (0, u, 0) + v(1, 0, u).
$$

(b) For the first parametrization we have $\varphi_u = (1, 0, v), \varphi_v = (0, 1, u), \varphi_{uu} = 0 = \varphi_{vv}, \varphi_{uv} = (0, 0, 1)$. A normal to the surface is $N = (y, x, -1)$. So we see that $m = \varphi_{uv} \cdot U \neq 0$ and the Gaussian curvature is not 0. Hence the surface is not developable.