MATH 150B

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Please explain or prove all your assertions and show your work. Please state all the definitions, propositions, theorems, lemmas that you use precisely.
Please make sure to review all definitions, statements of theorems, proofs done in class, practice problems for the midterms and homework problems.
There will be at most 9 questions on the final: 1 about definitions, 1 or 2 proofs, and at most 6 problems.
Good luck!

(1) Prove that holonomy along a curve does not depend on the choice of parallel vector field. Does it depend on the choice of frame field?

(2) In the Poincaré half plane, consider the frame field $E_1 := vU_1, E_2 := vU_2$.
   (a) Compute the dual 1-forms and the connection form for $\{E_1, E_2\}$ in terms of du, dv.
   (b) Compute the velocity and acceleration of the curve $\alpha(t) = (r\cos t, r\sin t), 0 < t < \pi, r$ in terms of $\{E_1, E_2\}$.
   (c) Compute the velocity and acceleration of the curve $\beta(t) = (ct, st)$ where t is such that $st > 0$, and c and s are constants such that $c^2 + s^2 = 1$.

(3) Compute the Gaussian curvature of the Poincaré half plane.

(4) Let f be a differentiable function of a real variable and let M be the image of the map

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \\ (u,v) & \longmapsto & (f(u)\cos v, f(u)\sin v, v) \end{array}$$

   (a) Describe the pull-back metric on $\mathbb{R}^2$.
   (b) Write a frame field for this metric on $\mathbb{R}^2$. Compute the dual forms, connection form and Gaussian curvature of this metric.

(5) A mapping of surfaces $F : M \to N$ is called conformal if there exists a function $\lambda : M \to \mathbb{R}_{>0}$ such that

$$||F_*(v_p)|| = \lambda(p)||v_p||$$

for all points $p \in M$ and all vectors $v_p \in T_pM$. The function λ is called the scale factor of F.

(a) Show that the mapping

$$\begin{array}{rcc} \mathbb{R}^2 & \twoheadrightarrow & M \quad \subset \quad \mathbb{R}^4 \\ (u,v) & \longmapsto & (\cosh u \cos v, \cosh u \sin v, \sinh u \cos v, \sinh u \sin v) \end{array}$$

is conformal onto its image M, hence regular.

(b) Find the Gaussian curvature of M.

(6) Let M be the image of the map

$$\begin{array}{rcc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^4 \\ (u,v) & \longmapsto & (u, v, uv, \frac{u^2-v^2}{2}). \end{array}$$

(a) Find the Gaussian curvature of M.

(b) Find the area of the region in M where $0 \le u, v \le 1$.