Algebraic Geometry 203A, Fall 2020.

Web page:

math.ucsd.edu/neizadi/203A/203A-2020/203A-2020.html

Algebraic Geometry is a very old subject. In some sense, we can say that it is as old as geometry or algebra. Algebraic Geometry is first "Geometry", meaning the goal is to study geometric objects, algebra is one of the tools we use to do so.

One could say that modern algebraic geometry started when geometers began to characterize geometric objects by the functions on them instead of their underlying sets. This is a dual point of view. This has revolutionized modern mathematics, also useful in other areas.

In this way, a manifold or variety or scheme or stack, ..., is given by the data of a set and a class of functions on that set. In fact the class of functions determines the set.

Algebro-geometric objects are geometric objects whose functions are obtained via algebraic operations, i.e., polynomials, or rational functions.

The simplest algebro-geometric objects are affine spaces, which in the dual point of view are polynomial rings.

Fix an algebraically closed field k.

For the time being, n-dimensional affine space is k^n. If $x_1, \dots, x_n$ are the coordinate functions on k^n, the associated ring is $k[x_1, \dots, x_n]$ = ring of polynomials in n variables with coefficients in k.

We define a topology on $k^n (= \mathbb{A}^n_k)$ using the functions:

<u>Def</u>: Given a subset $T \subset k[x_1, \dots, x_n]$ define $Z(T) := \{(a_1, \dots, a_n) \in k^n \mid P(a_1, \dots, a_n) = 0 \ \forall \ P \in T\}$

<u>Def</u>: The Zariski topology on k^n is the topology whose closed sets are the sets $Z(T)$ for all subsets $T \subset k[x_1, \dots, x_n]$.

<u>This is a topology</u>: <u>Verify</u>: (1) ϕ is closed
(2) k^n is closed

(3) arbitrary intersections of closed sets are closed

(4) finite unions of closed sets are closed.

<u>Proof:</u> (1) $\phi = Z(T) \qquad T = \{1\}$

(2) $k^n = Z(\{0\})$

(3) $P_1, P_2 \qquad Z(P_1) \cap Z(P_2) = Z(P_1, P_2)$

In general $\displaystyle\bigcap_{i \in I} Z(T_i) = Z\left(\bigcup_{i \in I} T_i\right)$

(4) $Z(P_1) \cup Z(P_2) = Z(P_1 P_2)$

$Z(T_1) \cup Z(T_2) = Z(T_1 T_2)$

where $T_1 T_2 := \{P_1 P_2 \mid P_1 \in T_1, P_2 \in T_2\}$.

Definition: Affine varieties are closed subsets of affine space. The ideal of an affine variety Y is

$$I(Y) := \left\{ P \in k[x_1, \dots, x_n] \;\middle|\; \begin{array}{l} P(a_1, \dots, a_n) = 0 \\ \forall\, (a_1, \dots, a_n) \in Y \end{array} \right\}$$

Verify: $I(Y)$ is an ideal in $k[x_1, \dots, x_n]$

The values of a polynomials $P \in k[x_1, \dots, x_n]$ on Y only depend on the class of P modulo $I(Y)$.

So we define the ring of regular functions on the coordinate ring of Y as:

$$A(Y) := k[x_1, \dots, x_n] \big/ I(Y) .$$

The next simplest algebro-geometric objects are projective spaces.

<u>Def</u>: $\mathbb{P}^n_k := \left(k^{n+1} \setminus \{0\}\right) / k^*$

where $k^* := k \setminus \{0\}$ acts via scalar multiplication on k^{n+1}. So $\mathbb{P}^n_k$ is the set of equivalence classes in $k^{n+1} \setminus \{0\}$ for the equivalence relation $(a_0, \dots, a_n) \sim (b_0, \dots, b_n)$

$\iff \exists \lambda \in k^*$ s.t. $a_i = \lambda b_i \;\; \forall i$.

So $\mathbb{P}^n_k$ is the set of lines through 0 in k^{n+1}.

Functions on $\mathbb{P}^n_k$? First place to look:

$$k[x_0, \dots, x_n]$$

This does not work: $n=1$ $\mathbb{P}^1_k$

e.g.: $P(x_0, x_1) = x_0^2 + x_1$ $(2,1) \in \mathbb{P}^1_k$. $k=\mathbb{R}$

$= (6,3)$

$P(2,1) = 4+1 = 5 \neq P(6,3) = 36+6$
$= 42$

The only well-defined functions are constants!

However, we can define the Zariski topology on $\mathbb{P}^n_k$ and the open sets of the Zariski topology will have lots of algebraic or regular functions on them.

We will define the closed sets of the Zariski topology as zero sets of "homogeneous" polynomials.

<u>Def</u>: A polynomial P is called homogeneous if it is a k-linear combination of monomials of the same degree. Equivalently, $\forall \lambda \in k$ and $(a_0, \dots, a_n) \in k^{n+1}$,

$$P(\lambda a_0, \dots, \lambda a_n) = \lambda^d P(a_0, \dots, a_n)$$

where d = degree of P. (= degree of the monomials in P)

<u>Note</u>: $Z(P)$ makes sense for P homogeneous $\subset \mathbb{P}^n_k$

If $(b_0, \dots, b_n) = \lambda (a_0, \dots, a_n)$ $\lambda \in k^*$

$P(b_0, \dots, b_n) = 0 \iff P(a_0, \dots, a_n) = 0$

<u>Definition</u>: Given $T \subset k[x_0, \dots, x_n]$ a set of homogeneous polynomials.

$$Z(T) := \{(a_0, \dots, a_n) \in \mathbb{P}^n \mid P(a_0, \dots, a_n) = 0 \ \forall P \in T\}.$$

<u>Def</u>: The Zariski topology on $\mathbb{P}^n$ is the topology with closed sets $Z(T)$ for all $T \subset k[x_0, \dots, x_n]$ subsets of homogeneous polynomials.

Verify: (1) $\phi = Z(x_0, x_1, \ldots, x_n)$

(2) $\mathbb{P}^n = Z(0)$

(3) $\bigcap_{i \in I} Z(T_i) = Z\left(\bigcup_{i \in I} T_i\right)$

(4) $Z(T_1) \cup Z(T_2) = Z(T_1 T_2)$

(if two polynomials are homogeneous, so is their product).