<u>Definition</u>: A projective variety is the set of zeros of a set of homogeneous polynomials in $\mathbb{P}^n_k$.

By analogy with the affine case, given

$Y = Z(T)$, $T \subset k[x_0, \dots, x_n]$ set of homogeneous polynomials

we define $I(Y) :=$ ideal generated by the set of homogeneous polynomials vanishing on Y.

<u>Fact</u>: $I(Y)$ is then a homogeneous ideal.

<u>Def</u>: An ideal $I \subset k[x_0, \dots, x_n]$ is called homogeneous if it can be generated by homogeneous polynomials.

Equivalently, if $I_d :=$ subvector space of I generated by polynomials of degree d.

then $I = \bigoplus_{d \geq 0} I_d$

Equivalently, $\forall\ P \in I$, if $P = \sum_{0 \leq i \leq d} P_i$ where P_i is the homogeneous component of P of degree i, then $\forall i$, $P_i \in I$.

<u>Def</u>: For a projective variety Y, its homogeneous coordinate ring is $S(Y) := k[x_0, \dots, x_n] / I(Y)$

Given an integer n, $\forall\ i \in \{0, \dots, n\}$, we have injective maps $\varphi_i : \mathbb{A}^n_k \hookrightarrow \mathbb{P}^n_k$

$$(y_1, \dots, y_n) \longmapsto \underset{\substack{\uparrow \\ i\text{-th spot.}}}{(y_1, \dots, 1, \dots, y_n)} = (y_1, \dots, y_{i-1}, 1, y_i, \dots, y_n)$$

image of φ_i? subset of $\mathbb{P}^n$ where $x_i = 1$

this just means $x_i \neq 0$ because we can multiply by elements of k^*.

So image is $V_i := \{(a_0, \dots, a_n) \mid a_i \neq 0\} \subset \mathbb{P}^n_k$

if $a_i \neq 0$, then $(a_0, \dots, a_n) \sim \left(\frac{a_0}{a_i}, \dots, 1, \dots, \frac{a_n}{a_i}\right)$

$$= \varphi_i\left(\frac{a_0}{a_i}, \dots, \frac{a_n}{a_i}\right)$$

injectivity: exercise.

<u>Note</u>, $\mathbb{P}^n_k = \bigcup_{i=0}^{n} V_i$ because $\forall (a_0, \dots, a_n) \in \mathbb{P}^n$
$\exists i$ s.t. $a_i \neq 0$

If we set $Y_i := Z(x_i)$,
then $Y_i = \mathbb{P}^n_k \setminus V_i$
Y_i is closed by definition, so V_i is open.

<u>Def</u>: Y_i is a coordinate hyperplane.

More generally, given a polynomial M of degree 1
$M \in k[y_1, \dots, y_n]$ $Z(M)$ is a hyperplane in $\mathbb{A}^n$
If $L \in k[x_0, \dots, x_n]$ is homogeneous of degree 1,
then $Z(L)$ is a hyperplane in $\mathbb{P}^n$.

$$L = \sum_{i=0}^{n} r_i x_i$$

For each i, L can restrict to $\mathbb{A}^n$ via φ_i as a hyperplane in $\mathbb{A}^n$

$$L \circ \varphi_i(b_1, \ldots, b_n) = L(b_1, \ldots, b_{i-1}, 1, b_i, \ldots, b_n)$$
$$= r_0 b_1 + \cdots + r_{i-2} b_{i-1} + r_{i-1} + r_i b_i + \cdots + r_n b_n$$

So L defines the polynomial

$$M := r_0 y_1 + \cdots + r_{i-2} y_{i-1} + r_{i-1} + r_i y_i + \cdots + r_n y_n$$

on $\mathbb{A}^n$ via φ_i

Homogenization and dehomogenization:

e.g.: $\mathbb{P}^2$ coordinates (X, Y, Z)

$$\mathbb{A}^2 \hookrightarrow \mathbb{P}^2 \qquad \varphi_2 : \mathbb{A}^2 \hookrightarrow \mathbb{P}^2$$

coordinates (x, y)

$$(x, y) \longmapsto (x, y, 1)$$

$$\mathbb{A}^2 \overset{?}{\longleftarrow} U_2 \subset \mathbb{P}^2$$

$$U_2 = \{Z \neq 0\}$$

$$\left(\frac{X}{Z}, \frac{Y}{Z}\right) \overset{\varphi_2^{-1}}{\longleftarrow\!\shortmid} (X, Y, Z) = \left(\frac{X}{Z}, \frac{Y}{Z}, 1\right)$$

Given a polynomial $P(x, y)$, we can homogenize it, and obtain a homogeneous polynomial in 3 variables.

This will be $Z^d P\left(\frac{X}{Z}, \frac{Y}{Z}\right)$ where $d = \text{degree}(P)$

<u>example</u>: $P = x^2 + y^2 + 1$

$$Z^2 P\left(\frac{X}{Z}, \frac{Y}{Z}\right) = Z^2\left(\frac{X^2}{Z^2} + \frac{Y^2}{Z^2} + 1\right)$$
$$= X^2 + Y^2 + Z^2$$

<u>example</u>: $x^3 + yx^5 - y + 1 + xy$

$$\rightsquigarrow \quad Z^3 X^3 + Y X^5 - Z^5 Y + Z^6 + Z^4 XY$$

Given a homogeneous polynomial $Q \in k[X, Y, Z]$, we can dehomogenize it to obtain $P \in k[x, y]$ by setting $Z = 1$, $X = x$, $Y = y$:

$$X^2 + Y^2 + Z^2 \rightsquigarrow x^2 + y^2 + 1$$
$$Z^3 X^3 + Y X^5 - Z^5 Y + Z^6 + Z^4 XY \rightsquigarrow x^3 + yx^5 - y + 1 + xy$$

$$k[y_1,\dots,y_n] \xrightarrow{H} k[x_0,\dots,x_n]$$

homogenize using φ_i

$d = \deg(P)$

$$P(y_1,\dots,y_n) \rightsquigarrow x_i^d\, P\left(\frac{x_0}{x_i},\dots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\dots,\frac{x_n}{x_i}\right)$$

$$k[x_0,\dots,x_n] \xrightarrow{D} k[y_1,\dots,y_n]$$

dehomogenize.

$$Q(x_0,\dots,x_n) \longmapsto Q(y_1,\dots,y_{i-1},1,y_i,\dots,y_n)$$

What we saw was that $D \circ H = \mathrm{Id}$ from $k[y_1,\dots,y_n]$ to itself.

What about $H \circ D$? $k[x_0,\dots,x_n] \longrightarrow k[x_0,\dots,x_n]$

$$Q(x_0,\dots,x_n) \rightsquigarrow P(y_1,\dots,y_n) := Q(y_1,\dots,1,\dots,y_n)$$

example: $X_0^2X_1^2 + X_1^2X_2^2 - X_1X_2^3 = X_1(X_0^2X_1 + X_1X_2^2 - X_2^3)$

dehomogenize w.r.t. X_1 : $\rightsquigarrow X_1 = 1$

$$y_1^2 + y_2^2 - y_2^3$$

homogenize: $\left(\left(\frac{X_0}{X_1}\right)^2 + \left(\frac{X_2}{X_1}\right)^2 - \left(\frac{X_2}{X_1}\right)^3\right) X_1^3$

$$X_1X_0^2 + X_2^2X_1 - X_2^3 = \frac{X_0^2X_1^2 + X_1^2X_2^2 - X_1X_2^3}{X_1}$$

$\leadsto$ $H \circ D$ is not the identity!

For an affine variety Y, the Zariski topology induces a topology on Y.

<u>Def.</u>: A quasi-affine variety is an open subset

of an affine variety.

<u>Def.</u>: A quasi-projective variety is an open subset of a projective variety.