Recall: k algebraically closed.

Nullstellensatz: There is a one-to-one inclusion reversing correspondence between affine varieties or algebraic subsets of $k^n = \mathbb{A}^n$ and radical ideals in $A := k[y_1, \dots, y_n]$ given by

$$Y \longmapsto I(Y)$$

$$Z(I) \longleftarrow I$$

Homogeneous Nullstellensatz: There is a one-to-one inclusion reversing correspondence between projective varieties or algebraic subsets of $\mathbb{P}^n$ and homogeneous radical ideals other than $\langle x_0, \dots, x_n \rangle$ in $S := k[x_0, \dots, x_n]$ given by $Y \longmapsto I(Y)$, $Z(I) \longleftarrow I$

where $Z(I)$ is the set of common zeros of the homogeneous elements of I.

<u>Dimension</u>: $\dim \mathbb{A}^n = \dim k^n = n$

$P = y_n$ $\qquad Z(P) \subset \mathbb{A}^n = k^n$

$\qquad\qquad\qquad \cong k^{n-1}$

$P = y_0 + y_1 - y_n$ $\qquad Z(P)$ has dim. $n-1$

So we define dimension in terms of descending chains of closed subsets.

For this to make sense, we need the notion of irreducibility.

Definition: A topological space Y is called irreducible if $Y \neq \emptyset$ and $\forall$ Y_1, Y_2 non-empty closed subsets of Y.

$$Y_1 \cup Y_2 = Y \Rightarrow Y_1 = Y \text{ or } Y_2 = Y.$$

Definition: A topological space is called Noetherian if it satisfies the descending chain condition for closed subsets, i.e., $\forall$ sequence

$$Y_1 \supseteq Y_2 \supseteq \dots \supseteq Y_n \supseteq \cdots$$

of closed subsets, $\exists\, n$ s.t. $\forall\, r \geq n$, $Y_r = Y_n$.

Def: The dimension of a topological space Y

is $\dim Y := \sup\{n \mid \exists$ chain of distinct irreducible closed subsets $Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_n\}$

In k^n, we have a chain

$$k^n = Y_0 \supsetneq Y_1 \supsetneq Y_2 \supsetneq \cdots \supsetneq \{0\}$$

$$Y_1 = k^{n-1} = Z(y_n), \quad Y_2 = Z(y_{n-1}, y_n), \quad \{0\} = Z(y_1, \ldots, y_n)$$

Theorem: (1) If $Y \subset \mathbb{A}^n$ is an irreducible affine variety, then $\dim Y = \dim A(Y) = \operatorname{trdeg}_k K(A(Y))$, where $K(A(Y)) = K(Y)$.

(2) If $Y \subset \mathbb{P}^n$ is an irreducible projective variety, then $\dim Y = \dim S(Y) - 1$

<u>Lemma:</u> $Y \subset \mathbb{A}^n$ or $\mathbb{P}^n$ affine or projective variety.

Y is irreducible $\iff$ $I(Y)$ is prime.

<u>Proof:</u> Assume $I(Y)$ is prime.

If $Y = Y_1 \cup Y_2$, then $I(Y) = I(Y_1) \cap I(Y_2)$

(Nullstellensatz) $= \sqrt{I(Y_1) \cdot I(Y_2)}$

More generally, given I s.t. $Z(I) = Y$, then

$I(Y) = \sqrt{I}$ by Nullstellensatz.

There is exactly one radical ideal whose zero set is Y, by Nullstellensatz and that is $I(Y)$.

$I(Y_1) \cdot I(Y_2) \subset I(Y) \Longrightarrow I(Y_1) \subset I(Y)$ or $I(Y_2) \subset I(Y)$

because $I(Y)$ is prime.

$\Longleftrightarrow \quad Y_1 \supset Y \quad \text{or} \quad Y_2 \supset Y$

$\Longleftrightarrow \quad Y_1 = Y \quad \text{or} \quad Y_2 = Y$

Assume Y is irreducible.

If $ab \in I(Y)$, then $Z(ab) = Z(a) \cup Z(b)$

$$\supset Z(I(Y)) = Y$$

$\Rightarrow \quad (Z(a) \cap Y) \cup (Z(b) \cap Y) = Y$

$\Rightarrow \quad Z(a) \cap Y = Y \quad \text{or} \quad Z(b) \cap Y = Y$

$\Rightarrow \quad Z(a) \supset Y \quad \text{or} \quad Z(b) \supset Y$

$\Rightarrow \quad a \in I(Y) \quad \text{or} \quad b \in I(Y) \qquad \square$

Dimension of a commutative ring:

<u>Def:</u> R a commutative ring. $\mathfrak{p} \subset R$ prime ideal.
The height of $\mathfrak{p}$ is

$$\operatorname{height}(\mathfrak{p}) := \sup\left\{ n \;\middle|\; \begin{array}{c} \exists\ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p} \\ \mathfrak{p}_i \text{ all prime} \end{array} \right\}$$

The Krull dimension of R is

$$\begin{aligned} \dim R &:= \sup\{ \operatorname{height}(\mathfrak{p}) \mid \mathfrak{p} \subset R \text{ prime} \} \\ &= \sup\left\{ n \;\middle|\; \begin{array}{c} \exists\ \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \subset R \\ \mathfrak{p}_i \text{ prime} \end{array} \right\} \end{aligned}$$

<u>Prop. (1.8A):</u> R integral domain, finitely generated over k, then, $\forall\ \mathfrak{p} \subset R$ prime ideal:

$$\dim(R/\mathfrak{p}) + \operatorname{height}(\mathfrak{p}) = \dim(R)$$

Proof of $\dim Y = \dim A(Y)$: for Y affine irred.

$$Y \supseteq Y_0 \supsetneq Y_1 \supsetneq Y_2 \supsetneq \cdots \supsetneq Y_n \neq \emptyset$$

$$\Longleftrightarrow \quad I(Y) \subseteq I(Y_0) \subsetneq I(Y_1) \subsetneq \cdots \subsetneq I(Y_n) \subsetneq A$$

$$\Longleftrightarrow \quad 0 \subseteq \frac{I(Y_0)}{I(Y)} \subsetneq \frac{I(Y_1)}{I(Y)} \subsetneq \cdots \subsetneq \frac{I(Y_n)}{I(Y)} \subseteq A(Y)$$

$$\Rightarrow \quad \dim Y = \dim A(Y) \qquad \square.$$

__Theorem:__ $\dim A(Y) = \mathrm{trdeg}_k K(A(Y))$ from commutative algebra (see references)

__example:__ $Y \subset \mathbb{A}^n$ is a hypersurface if $Y = Z(f)$ for $f \in A$. $Y = Z(\langle f \rangle)$

Y irreducible $\Longleftrightarrow$ $I(Y) = \sqrt{\langle f \rangle}$ is prime.

$f = g_1^{r_1} \cdots g_m^{r_m}$ g_i irreducible polynomial.

then $\sqrt{\langle f \rangle} = \langle g_1 \cdots g_m \rangle$

$\sqrt{(f)}$ is prime $\iff$ $m=1$

$\iff$ $f =$ prime power.

$\dim Y = \dim A(Y) = \dim A/\langle g_1 \rangle$

$= \dim A - \text{height}\langle g_1 \rangle$ by prop. 1.8(A)

$= n - 1$ By Theorem 1.11(A):

If $f \in A$ is not 0, then every minimal prime ideal containing f has height 1.

recall $\mathbb{P}^n = \bigcup_{i=0}^{n} U_i$ $\quad U_i = \{x_i \neq 0\}$

$\phi \neq Y = \bigcup_{i=0}^{n} Y \cap U_i$ $\quad \exists i$ s.t. $Y \cap U_i \neq \phi$

Lemma: $\dim Y = \dim (Y \cap U_i)$

Proof of lemma: Fact: (homework) any non-empty open subset of an irreducible topological space is dense.

assume $i=0$, i.e., $Y \cap U_0 \neq \phi$

given a chain $\phi \neq Y_0 \subsetneq Y_1 \subsetneq \dots \subsetneq Y_n \subseteq Y$

we obtain a chain $\phi \neq Y_0 \cap U_0 \subsetneq Y_1 \cap U_0 \subsetneq \dots \subsetneq Y_n \cap U_0$

why are the inclusions strict: