Proof of: If $Y \subset \mathbb{P}^n$ closed and irreducible, then $\dim Y = \dim S(Y) - 1$.

Given a chain of irreducible closed subsets of Y:

$$Y = Y_1 \supsetneq Y_2 \supsetneq \cdots \supsetneq Y_r \neq \phi,$$

the corresponding radical homogeneous ideals are:

$$I(Y) = I(Y_1) \subsetneq I(Y_2) \subsetneq \cdots \subsetneq I(Y_r) \underset{Y_r \neq \phi}{\subsetneq} \underset{\langle x_0, \dots, x_n \rangle}{\underset{\shortparallel}{S_+}} \subsetneq S$$

these are all prime.

We can go back to the zero sets Y_i, given homogeneous prime ideals $\mathfrak{p}_i \supseteq I(Y)$: $Y_i = Z(\mathfrak{p}_i)$.

$\Longleftrightarrow$ chain of prime ideals $\bar{\mathfrak{p}}_i = \mathfrak{p}_i / I(Y) \subset S(Y)$
$\subseteq S_+(Y) =$ image of S_+ in $S(Y)$

$$\Rightarrow \quad \dim(Y) = \text{height}(S_+(Y)) = \dim(S(Y)) - 1 . \quad \square.$$

<u>Now back to:</u> $\quad \mathbb{P}^n = \bigcup_{i=0}^n V_i \qquad Y = \bigcup_{i=0}^n Y \cap V_i$

<u>proof of:</u> If $Y \cap V_i \neq \phi$, then $\dim Y = \dim(Y \cap V_i)$.

Assume $i = 0$ WLOG.

we know $\dim Y = \dim S(Y) - 1 = n+1 - \text{height}\, I(Y) - 1$

$$\left(\text{height}\, I(Y) + \dim S(Y) = \dim S = n+1 \right)$$

$$\Rightarrow \dim Y = n - \text{height}\, I(Y).$$

$Y \cap V_0 \subset V_0 =$ image of $k^n \simeq k^n$

$$k^n \hookrightarrow \mathbb{P}^n$$

$$(y_1, \dots, y_n) \longmapsto (1, y_1, \dots, y_n)$$

$$\Rightarrow \dim Y \cap V_0 = \dim A(Y \cap V_0) = n - \text{height}\, I(Y \cap V_0)$$

So: $\dim Y = \dim (Y \cap V_0) \iff \text{height } I(Y) = \text{height } I(Y \cap V_0)$

note: $I(Y) \subset S = k[x_0, \dots, x_n]$, $I(Y \cap V_0) \subset A = k[y_1, \dots, y_n]$

recall: $h: A \to S$ homogenization

$$P \mapsto P_h := x_0^d P\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$$

$P = P(y_1, \dots, y_n)$, $d =$ degree of P

dehomogenization: $dh: S \to A$

$$Q \mapsto Q_{dh} := Q(1, y_1, \dots, y_n)$$

$Q = Q(x_0, \dots, x_n)$.

Let $I(Y) \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1, \dots, \supsetneq \mathfrak{p}_n$

be a chain of prime ideals in $I(Y)$

$\rightsquigarrow \quad I(Y_0) \supseteq \mathfrak{p}_{0,dh} \supseteq \mathfrak{p}_{1,dh} \supseteq \dots \supseteq \mathfrak{p}_{n,dh}$

We need to verify that the inclusions are strict:

If, for some j, $\mathfrak{p}_{j,dh} = \mathfrak{p}_{j+1,dh}$, then

for every $a_{dh} \in \mathfrak{p}_{j,dh}$ ($a \in \mathfrak{p}_j \setminus \mathfrak{p}_{j+1}$), $a_{dh} \in \mathfrak{p}_{j+1,dh}$

(i.e., $\exists$ $b \in \mathfrak{p}_{j+1}$, s.t. $a_{dh} = b_{dh}$

$\Leftrightarrow$ $\exists$ $m \geq 0$ s.t. $a = x_0^m b$ or $b = x_0^m a$

$\Downarrow$ $a \in \mathfrak{p}_{j+1}$ not allowed

$\Downarrow$ $a \notin \mathfrak{p}_{j+1}$ $x_0 \in \mathfrak{p}_{j+1}$

$x_0 \in \mathfrak{p}_{j+1} \subseteq I(Y)$ $\Rightarrow$ x_0 vanishes on Y

$\Rightarrow$ $Y \subset \mathbb{P}^n \setminus V_0$

$\Rightarrow$ $Y \cap V_0 = \phi$.

not possible either.

So if $a \notin \mathfrak{p}_{j+1}$, then $a_{dh} \notin \mathfrak{p}_{j+1}$ $(a \in \mathfrak{p}_j)$. $\Rightarrow$ $\mathfrak{p}_{j+1} \subsetneq \mathfrak{p}_j$ $\forall j$.

$$\Rightarrow \quad \text{height } I(Y) \leq \text{height } I(Y \cap V_0)$$

For the reverse inequality, we start with a chain of prime ideals

$$I(Y \cap V_0) \supseteq \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_m$$

$$\Rightarrow \quad I(Y) \supseteq \mathfrak{p}_{0,h} \supseteq \mathfrak{p}_{1,h} \supseteq \cdots \supseteq \mathfrak{p}_{m,h}$$

the inclusions are strict because: $(P_h)_{dh} = P$

$$\Rightarrow \quad \mathfrak{p}_i = (\mathfrak{p}_{i,h})_{dh}$$

$$\Rightarrow \quad \text{height } I(Y) \geq \text{height } I(Y \cap V_0) \quad \square$$

<u>Proposition</u>: In a noetherian topological space X, any closed subset Y is a finite union $Y_1 \cup \cdots \cup Y_n$ of irreducible closed subsets. If $\forall\, i \neq j$, $Y_i \not\subset Y_j$,

the decomposition is unique up to reindexing.

<u>Def</u>: The Y_i as above are called the irreducible components of Y.

<u>Proof</u>: <u>Existence</u>: Put

$$\mathfrak{S} := \left\{ \begin{array}{l} \text{non-empty closed subsets that are } \underline{\text{not}} \text{ finite} \\ \text{unions of irreducible closed subsets} \end{array} \right\}$$

If $\mathfrak{S} \neq \phi$, then every descending chain of elements of $\mathfrak{S}$ has a minimal element because X is noetherian. So, by Zorn's lemma, $\mathfrak{S}$ has a minimal element, say Y. Since $Y \in \mathfrak{S}$, then Y is not irreducible. So $Y = Y_1 \cup Y_2$ where Y_1, Y_2 are proper closed subsets of Y. Since Y_i are strictly smaller than Y and

Y is minimal in $\mathfrak{S}$, $Y_1, Y_2 \notin \mathfrak{S}$.

$\Rightarrow$ Y_1 and Y_2 are finite unions of irreducible closed subsets.

$\Rightarrow$ Y is a finite union of irred. closed subsets $\Rightarrow Y \notin \mathfrak{S}$ contradiction!

<u>Uniqueness:</u> Suppose:

$$Y = Y_1 \cup \cdots \cup Y_r = Y_1' \cup \cdots \cup Y_s'$$

Since Y_r is irreducible and contained in $Y_1' \cup \cdots \cup Y_s'$, $\exists\, j$ s.t. $Y_r \subset Y_j'$. Similarly, $\exists\, i$ s.t. $Y_j' \subset Y_i$.

$$\Rightarrow Y_r \subset Y_j' \subset Y_i \Rightarrow Y_r = Y_i$$
$$\Rightarrow Y_r = Y_i = Y_j'$$

Remove Y_r from both sides to obtain unions with one less closed set in them and repeat to obtain $r = s$

and $\forall i, \exists j : Y_i = Y'_j$ $\square$

<u>A few words about cones:</u>

The affine and projective cones over a projective variety: Let $\theta : \mathbb{A}^{n+1} \setminus \{0\} \twoheadrightarrow \mathbb{P}^n$ be the quotient map. For any projective variety $Y \subset \mathbb{P}^n$, define the affine cone over Y to be

$$C(Y) := \theta^{-1}(Y) \cup \{0\} \subseteq \mathbb{A}^{n+1}!$$

By definition, the ideal of $C(Y)$ as an affine variety is equal to the homogeneous ideal of Y as a projective variety. Therefore $\dim C(Y) = \dim Y + 1$

Now embed $\mathbb{A}^{n+1} \hookrightarrow \mathbb{P}^{n+1}$ as, e.g., the open set $U_{x_{n+1}}$

The projective cone over Y is the closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbb{P}^{n+1} \hookleftarrow V_{n+1} = \mathbb{A}^{n+1}$.

The ideal of $\overline{C(Y)}$ is the homogenization of $I(Y)$ with respect to the extra variable x_{n+1}.