The projective cone over Y is the closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbb{P}^{n+1} \hookleftarrow V_{n+1} = \mathbb{A}^{n+1}$.

The ideal of $\overline{C(Y)}$ is the homogenization of $I(Y)$ with respect to the extra variable x_{n+1}.

<u>Note:</u> this is "the same" ideal, but thought of as an ideal in $k[x_0, \ldots, x_{n+1}]$.

Now we start talking about sheaves:
As mentioned before, sheaves are a generalization of functions, or a localized way of thinking about functions.

Presheaves:

Definition: Let X be a topological space.
A presheaf $\mathcal{F}$ (of sets) on X is the data,
(1) for each open set $U \subset X$, of a set $\mathcal{F}(U)$
(2) for every inclusion of open sets $V \subset U$, of a "restriction" map $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$,

such that:
(1) $\mathcal{F}(\emptyset) = \{\emptyset\}$ is the set with one element,
(2) $\rho_{UU} = Id_U$ the identity map of $\mathcal{F}(U)$,
(3) for all inclusions of open sets $W \subset V \subset U$, we have $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Note: A presheaf is a contravariant functor from the category of open sets of X to the category of sets.

We can change the target category to Abelian groups, rings, modules over a fixed ring, etc.

We would change our maps of sets to homomorphisms of abelian groups, rings, modules, etc.

$\{\phi\}$ would be replaced by a final object in the relevant category, which is $\{0\}$ for ablian groups, rings or modules.

Example: Constant presheaves: $\mathcal{F}(U) = S$ for a fixed set S, $\forall\, U \neq \phi$.

We build the important properties of functions into the definition of sheaves:

<u>Notation:</u> From now on, we denote $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ by $|_V$.

<u>Definition:</u> A presheaf $\mathcal{F}$ on X is a sheaf if it satisfies the following:

(1) for all U and all open coverings $U = \bigcup_{i \in I} V_i$, if $s, t \in \mathcal{F}(U)$ are "sections of $\mathcal{F}$ over U" s.t.

$$s|_{V_i} = t|_{V_i} \quad \forall\, i \in I, \text{ then } s = t,$$

(2) for all U and all open coverings $U = \bigcup_{i \in I} V_i$, and all collections $\{ s_i \in \mathcal{F}(V_i) \mid i \in I \}$ s.t.

$$s_i\big|_{V_i\cap V_j} = s_j\big|_{V_i\cap V_j}, \ \exists\ s\in\mathcal{F}(V) \text{ such that}$$

$$s\big|_{V_i} = s_i \quad \forall\, i\in I.$$

<u>Note</u>: We can combine these into one condition by requiring s to be unique in (2).

<u>Note</u>: the constant presheaf is NOT a sheaf.

<u>Examples of sheaves</u>: (1) X, S two topological spaces

$\mathcal{F} :=$ sheaf of continuous functions from X to S.

$$\mathcal{F}(V) = \{\text{continuous functions from } V \text{ to } S\}.$$

If S has the discrete topology, then $\mathcal{F}$ is the constant "sheaf" on X with values in S.

(2) Sheaves of C^∞ functions or differential forms on C^∞ manifolds.

(3) Sheaves of holomorphic functions on complex analytic spaces, or real analytic functions on real analytic spaces.

(4) $\pi : E \twoheadrightarrow X$ continuous. Sheaf of continuous sections of π:

$$\forall \; U \subset X \qquad \mathcal{F}(U) = \{ s : U \to E \mid \pi \circ s = Id_U \}$$

continuous.

We want to define algebro-geometric sheaves with "algebraic" functions. For this, we first "expand" on affine and projective varieties.

Back to Nullstellensatz:

Statement: k algebraically closed. For all ideals $I \subset k[x_1, \dots, x_n]$, $I(V(I)) = \sqrt{I}$.

Proof: Clearly $\sqrt{I} \subset I(V(I))$.

For the inclusion in the other direction, we show that if $f \notin \sqrt{I}$, then $f \notin I(V(I))$.

We need to find a point $(a_1, \dots, a_n) \in V(I)$ s.t. $f(a_1, \dots, a_n) \neq 0$.

we know $\sqrt{I} = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}$.

So $\exists \; \mathfrak{p} \supset I$ s.t. $f \notin \mathfrak{p}$.

Put $R := A/\mathfrak{p} := k[x_1, \dots, x_m]/\mathfrak{p}$.

This is an integral domain, so it has a field of fractions $K(R)$. $R \hookrightarrow K(R)$

Let $g \in R$ be the image of f: $g \neq 0$.

Let $R[g^{-1}] \subset K(R)$ be the subring generated by R and g^{-1}.

Choose a maximal ideal $\mathfrak{m} \subset R[g^{-1}]$, then

$R \cap \mathfrak{m}$ does not contain g because g is invertible in $R[g^{-1}]$.

$R \cap m \subset R$ is a prime ideal.

In fact $R\cap m$ is maximal in R because

$$R/_{R\cap m} \cong R[g^{-1}]/_{m}$$ is a field.

In fact, we have more: the inverse image of

$R\cap m$ in A is also maximal:

$$\begin{tikzcd}
 & & k \arrow[d, hook] & & & \\
k \arrow[r, hook] & A \arrow[r, two heads, "\pi"] \arrow[d] & R \arrow[r, hook] \arrow[dr] & R[g^{-1}] \arrow[r, two heads] & R[g^{-1}]/_{m} = \text{field} & \text{finite ext. of } k \\
 & A/_{\pi^{-1}(R\cap m)} \arrow[rr, two heads, "\cong"] & & \underbrace{R/_{R\cap m}}_{} \arrow[ur, hook, "\cong"'] \arrow[rr, "\mathrm{id}"] & & = k \\
 & \cong k & & k \arrow[u, hook] & &
\end{tikzcd}$$

$1 \mapsto 1$.

The quotient of any finitely generated algebra over a field k is a finite extension of k.

Let $a_1, \dots, a_m$ be the images of $x_1, \dots, x_m$ in

$$k = A/_{\pi^{-1}(R \cap m)} := R/_{R \cap m}$$

$(a_1, \dots, a_m) \in k^m = A^m$, then $f(a_1, \dots, a_m) \neq 0$

because g is the image of f and g is not zero in $R/_{R \cap m}$.

In fact $\pi^{-1}(R \cap m) = (x_1 - a_1, \dots, x_m - a_m)$.

because $x_i - a_i \longmapsto 0$ in $k = A/_{\pi^{-1}(R \cap m)}$

$$\Rightarrow (x_1 - a_1, \dots, x_m - a_m) \subset \pi^{-1}(R \cap m)$$

and $(x_1 - a_1, \dots, x_n - a_n)$ is maximal.

However $\pi^{-1}(R \cap m) = (x_1 - a_1, \dots, x_n - a_n) \supset p \supset I$

because p is the kernel of $A \twoheadrightarrow R$

$\Rightarrow (a_1, \dots, a_n) \in V(I)$

(any polynomial in I can be written as $\sum_{i=1}^{n} P_i (x_i - a_i)$)

$\Rightarrow f$ does not vanish on $V(I)$

$\Rightarrow f \notin I(V(I))$ $\square$

Note: We also proved that every maximal ideal of A is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $(a_1, \dots, a_n) \in k^n$.