We want to have a "coherent" theory of varieties. Such a theory should also work for fields that are <u>not</u> algebraically closed.

So, we expand affine space $\mathbb{A}^n = k^n$ by adding all the maximal ideals of $A = k[x_1, \dots, x_n]$.

This is still not enough:

We want to translate geometric constructions or procedures in terms of functions.

E.g. a map $\varphi : k^m \longrightarrow k^n$ corresponds to a homomorphism of rings $\varphi^* : k[y_1, \dots, y_n] \longrightarrow k[x_1, \dots, x_m]$

$$P \in k[y_1, \dots, y_n] \qquad P \mapsto \varphi^*(P) := P \circ \varphi$$

$P$ is a function $k^n \to k$. $\quad \mathfrak{m}$ = maximal ideal of $k[x_1, \dots, x_m]$

$$\mathfrak{m} \longmapsto \varphi(\mathfrak{m}) := (\varphi^*)^{-1}(\mathfrak{m})$$

<u>Problem:</u> In general, the inverse image of a maximal ideal by a homomorphism of rings is not necessarily maximal. But, it is prime!

So we define, given $k$:

<u>Def:</u> $\mathbb{A}^n_k$ affine $n$-space over $k$ is as a set

$$\operatorname{Spec} k[x_1, \dots, x_n] := \{\mathfrak{p} \mid \mathfrak{p} \subset k[x_1, \dots, x_n] \text{ prime ideal}\}$$

<u>Def:</u> $R$ a commutative ring with 1. The spectrum of $R$ as a set is

$$\operatorname{Spec} R := \{\mathfrak{p} \mid \mathfrak{p} \subset R \text{ prime ideal}\}$$

Def: $R$ a commutative ring with $1$. The affine $n$-space over $R$ as a set is

$$\mathbb{A}^n_R := \operatorname{Spec} R[x_1, \ldots, x_n].$$

Next, we need to define a topology on $\operatorname{Spec} R$.

For affine varieties, the closed sets of the topology were the sets of zeros of collections of polynomials.

Given $f \in k[y_1, \ldots, y_n]$

To say that $f(a_1, \ldots, a_n) = 0$

$$\Leftrightarrow \quad f \in (x_1 - a_1, \ldots, x_n - a_n)$$

So to say $f(p) = 0$ means $f \in p$.

For an ideal $I \subset k[x_1, \ldots, x_n]$, to say that the elements of $I$ vanish at $p$, means $I \subset p$.

<u>Def</u>: $R$ comm. ring with $1$. The Zariski topology on $\operatorname{Spec} R$ is the topology whose closed sets are the sets $V(I)$ where, for an ideal $I \subset R$,

$$V(I) := \{ p \mid p \supset I \} \subset \operatorname{Spec} R.$$

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Now we want to define functions on $\operatorname{Spec} R$. In fact we will define a sheaf of rings.

On affine space $\mathbb{A}^n_k$, we have the ring of functions: $k[x_1, \ldots, x_n]$, and $\mathbb{A}^n_k = \operatorname{Spec} k[x_1, \ldots, x_n]$

So the ring of functions on $\operatorname{Spec} R$ is $R$.

To define the sheaf, think of affine space.

Choose $f \in k[y_1, \dots, y_n]$ $\quad V(f) =$ set of zeros of $f$

$U_f := \mathbb{A}^n_k \setminus V(f)$ is a "basic" open set.

The ring of functions on this open set is

$$k[y_1, \dots, y_n]\left[\frac{1}{f}\right] \subset k(y_1, \dots, y_n)$$

or $$= k[y_1, \dots, y_n, X] \big/ (fX - 1)$$

In $\operatorname{Spec} R$, we have the "basic" open sets:

$$U_f := \operatorname{Spec} R \setminus V(f) \quad \text{for } f \in R$$
$$= \{ \mathfrak{p} \mid \mathfrak{p} \subset R,\ \mathfrak{p} \not\ni f \}$$

The ring of $U_f$ is $\mathcal{O}(U_f) := R[\frac{1}{f}]$

$$:= R[X]/(fX-1)$$

<u>Lemma:</u> Any open set in Spec R is a union of basic open sets, i.e., open sets of the form $U_f := \operatorname{Spec} R \setminus V(f)$ for $f \in R$. In other words, the basic open sets for a basis for the Zariski topology.

<u>Proof:</u> $U \subset \operatorname{Spec} R$ open. Then $U = \operatorname{Spec} R \setminus V(I)$ for some ideal $I$. Choose a set of generators $\{f_j\}$ for $I$. Then $V(I) = \bigcap_j V(f_j)$, and

$$U = \bigcup_j U_{f_j}$$

$\square$

Lemma: $\operatorname{Spec} R$ is quasi-compact.

Proof: Let $\operatorname{Spec} R = \bigcup_{i\in I} V_i$ be an open covering.

Then, for each $i$, $V_i$ has a covering $V_i = \bigcup_{j\in J_i} U_{f_{ij}}$

by basic open sets. So $\operatorname{Spec} R = \bigcup_{i,j} U_{f_{ij}}$

$$\Longleftrightarrow \quad \bigcap_{i,j} V(f_{ij}) = \phi$$

$$\| $$

$$V(\{f_{ij} \mid i,j\}) = V(\langle f_{ij} \mid i,j \rangle)$$

So there are NO prime ideals that contain

$\langle f_{ij} \mid i,j \rangle$ , so $\langle f_{ij} \mid i,j \rangle = R$ .

$$\Longleftrightarrow \quad 1 \in \langle f_{ij} \mid i,j \rangle$$

$$\Longleftrightarrow \quad \exists\, i_1, \dots, i_n,\ j_1, \dots, j_n,\ a_1, \dots, a_n \in R$$

$$\text{s.t.} \quad 1 = \sum_{l=1}^{n} a_l f_{i_l j_l}$$

$$\Longrightarrow \quad \langle f_{i_l j_l} \mid l = 1, \dots, n \rangle = R$$

$$\Longrightarrow \quad \bigcap_{1 \le l \le n} V(f_{i_l j_l}) = \emptyset \quad \Longleftrightarrow \quad \bigcup_{1 \le l \le n} U_{f_{i_l j_l}} = \operatorname{Spec} R$$

$$\Longrightarrow \quad \bigcup_{1 \le l \le n} V_{i_l} = \operatorname{Spec} R$$

$\square$

Good reference : Eisenbud-Harris : Geometry of schemes.

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To continue with our constructions, we need a few more concepts: co-limits or direct limits
limits or inverse limits.

<u>Def</u>: A <u>directed set</u> is a partially ordered set $(I, \leq)$ s.t. $\forall$ $i, j \in I$, $\exists$ $k \in I$ s.t. $k \geq i$ and $j$.

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Given an index set $I$ (this may or may not be a directed set, some authors require it and some don't)

and a collection of set $\{ S_i \mid i \in I \}$

we define:

<u>Def</u>: Given a collection of maps

$$\varphi_{ij} : S_i \longrightarrow S_j \quad \forall\ i \leq j$$

s.t. $\forall\ i \leq j \leq k \qquad \varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$.

we define the direct limit $\varinjlim S_i$

to be $\coprod_{i \in I} S_i \Big/ \sim$

where $\alpha_i \in S_i \sim \beta_j \in S_j$ if $\exists\ k \geq i, j$

s.t. $\varphi_{ik}(\alpha_i) = \varphi_{jk}(\beta_j)$.

<u>Def</u>: Given a collection of maps

$$\varphi_{ij} : S_j \longrightarrow S_i \qquad \forall \; i \leq j$$

s.t. $\forall \; i \leq j \leq k \qquad \varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$

we define the inverse limit $\varprojlim S_i$

as the subset of the direct product

$\prod_{i \in I} S_i$ of sequences $(\alpha_i)_{i \in I}$

s.t. $\forall \; i \leq j \qquad \varphi_{ij}(\alpha_j) = \alpha_i$