We need to define the ring of functions of an arbitrary open set $U \subset \operatorname{Spec} R$.

<u>Def</u>: The ring $\mathcal{O}(U)$ is by definition

$$\mathcal{O}(U) := \varprojlim_{U_f \subset U} R[f^{-1}]$$

Our index set here is the set of elements of R s.t. $U_f \subset U$ with partial order

$$f \leq g \iff U_f \subset U_g \iff \exists n \geq 1 \; g \mid f^n$$

$$\Updownarrow$$

$$\left(V(f) \supset V(g) \iff \sqrt{(f)} \subset \sqrt{(g)} \iff f \in \sqrt{(g)} \right)$$

<u>Exercise:</u> Verify that this defines a sheaf (see e.g. "Geometry of schemes" by Eisenbud–Harris)

We are going to see this is a sheaf in a different way which is a more general construction. For that we need:

Def: The stalk of a presheaf $\mathcal{F}$ on X at a point $x \in X$ is $\mathcal{F}_x := \varinjlim_{x \in U \subset X} \mathcal{F}(U)$

where the directed set for the direct limit is the set of open neighborhoods of x ordered by inclusion.

More concretely: $\mathcal{F}_x = \coprod_{x \in U \subset X} \mathcal{F}(U) \Big/ \sim$

$$= \left\{ (U, s) \,\middle|\, x \in U \subset X,\ s \in \mathcal{F}(U) \right\} \Big/ \sim$$

where $(U, s) \sim (V, t) \iff \exists\, W \subset U \cap V,\ x \in W$

s.t. $s|_W = t|_W$

Exercise: Sheaves are determined by their values on a basis of the topology, but presheaves are not. In a sense that we will make precise later, sheaves are determined by their stalks (⚠ source of mistakes!), but presheaves are not!

Given a presheaf, we can associate to it a unique sheaf, in a canonical way:

Def: The sheaf $\tilde{\mathcal{F}}$ associated to the presheaf $\mathcal{F}$ is the unique (up to isomorphism) sheaf with a morphism

$f: \mathcal{F} \longrightarrow \tilde{\mathcal{F}}$ such that, for any sheaf $\mathcal{G}$, any morphism of presheaves $\mathcal{F} \xrightarrow[f]{} \mathcal{G}$ factors uniquely through $\tilde{\mathcal{F}}$:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\varphi"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G} \\
\tilde{\mathcal{F}} \arrow[ur, dashed, "\exists!\,\tilde{\varphi}"'] &
\end{tikzcd}$$

(commutative diagram)

<u>Definition</u>: A morphism of presheaves $f: \mathcal{F} \to \mathcal{G}$ is the data of maps $\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ $\forall U \subset X$ s.t. $\forall\ V \subset U$ the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "f(U)"] \arrow[d, "\text{restriction for } \mathcal{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}(U) \arrow[d, "\text{restriction for } \mathcal{G}"] \\
\mathcal{F}(V) \arrow[r, "f(V)"] & \mathcal{G}(V)
\end{tikzcd}$$

A morphism of sheaves $\mathcal{F} \to \mathcal{G}$ is a morphism of the underlying presheaves.

One construction of the sheaf $\tilde{\mathcal{F}}$ associated to the presheaf $\mathcal{F}$ uses the "espace étalé" of a presheaf.

<u>Def</u>: The espace étalé of the presheaf $\mathcal{F}$ on X is the set

$$\begin{array}{ccc} \overline{\mathcal{F}} := \coprod_{x \in X} \mathcal{F}_x & \text{with a topology (defined below)} \\ \text{natural projection} \quad \pi \downarrow \qquad\qquad \downarrow & \\ X \ni x & \end{array}$$

A section $s \in \mathcal{F}(U)$ (the elements of $\mathcal{F}(U)$ are the sections of $\mathcal{F}$ over U) defines a section of π by sending $x \in U$ to the germ of s at x, meaning the equivalence class $s(x)$ of (U,s) in the stalk $\mathcal{F}_x$.

$$\begin{tikzcd}
\tilde{\mathcal{F}} \arrow[d] & \supset & \coprod_{x\in U} \mathcal{F}_x & \ni s(x) \\
X & \supset & U \arrow[u, bend right, "s"'] & \ni x \arrow[u, |->]
\end{tikzcd}$$

We endow $\tilde{\mathcal{F}}$ with the topology whose open sets are unions of open sets of the form $s(U)$ for $U \subset X$ and $s \in \mathcal{F}(U)$.

The sheaf $\tilde{\mathcal{F}}$ can be defined as the sheaf of (germs of) continuous sections of π.

<u>More concretely:</u> $U \subset X$ $\quad \tilde{\mathcal{F}}(U)$?

$$\tilde{\mathcal{F}}(U) = \{ f : U \to \tilde{\mathcal{F}} \mid f \text{ continuous} \,\&\, \pi \circ f = \mathrm{Id}_U \}$$
$$= \{ f : U \to \coprod_{x \in U} \mathcal{F}_x \mid f \text{ continuous} \,\&\, \forall x \in U \;\; f(x) \in \mathcal{F}_x \}$$

Understanding the continuity:

$\forall\, U \subset X, \quad \forall\, V \underset{\text{open}}{\subset} \bar{\mathcal{F}} \qquad f^{-1}(V) \subset U$ open

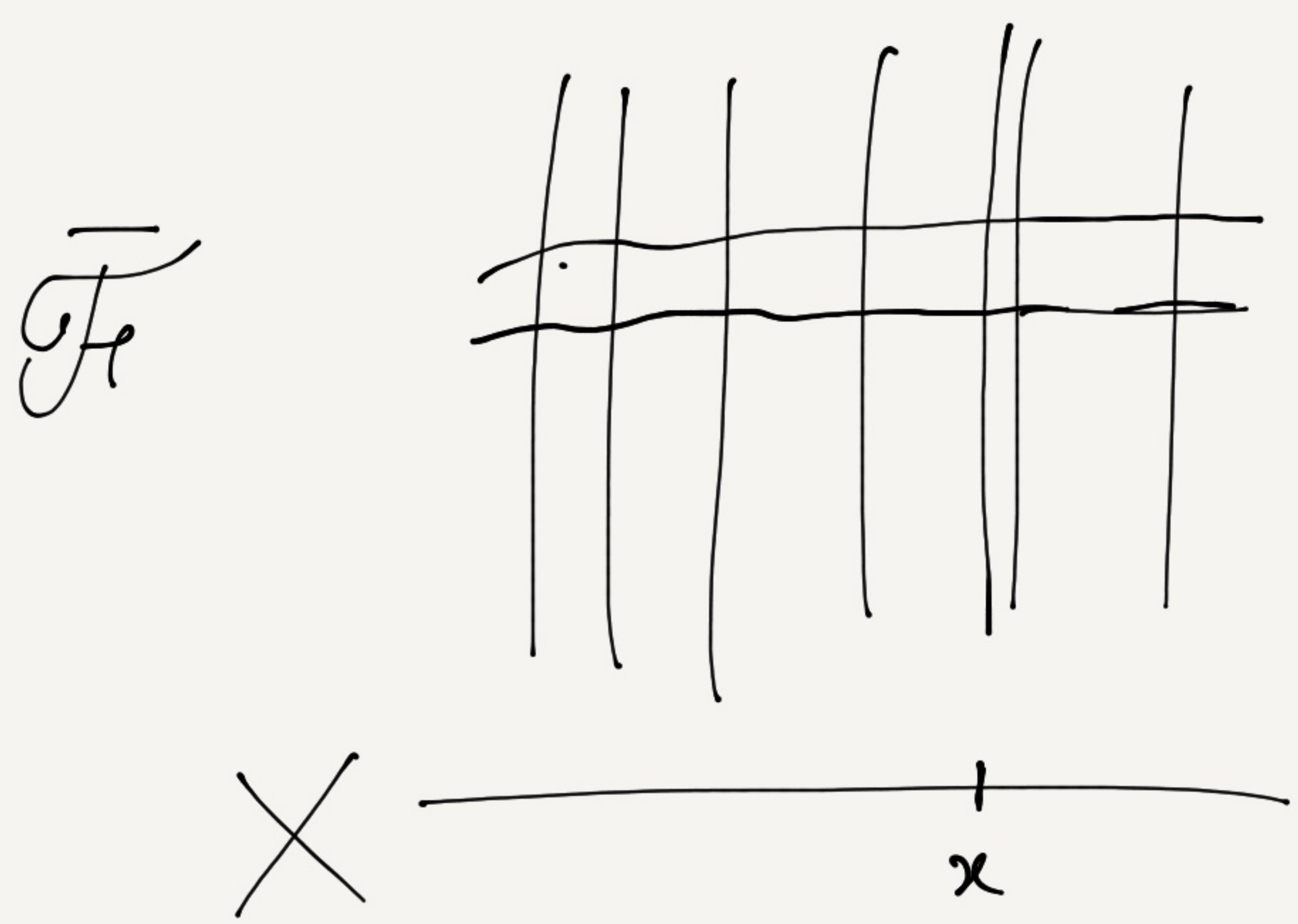
$V =$ union of images $s(W)$ for $W \underset{\text{open}}{\subset} X$ and $s \in \mathcal{F}(W)$.

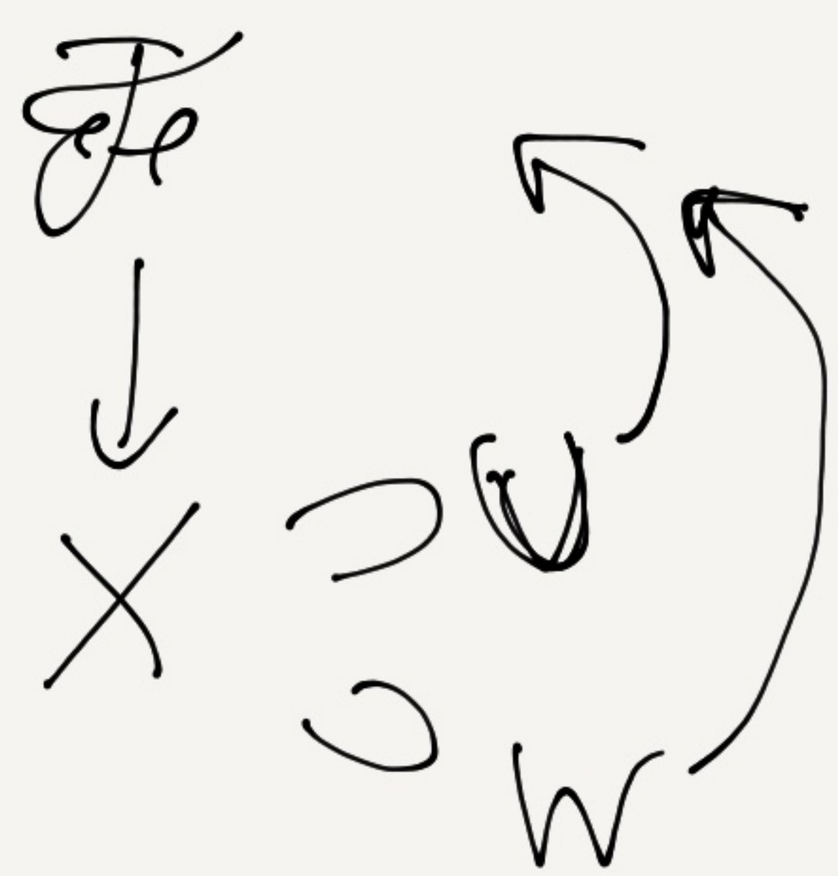
$\iff \quad \forall\, U \subset X, \quad \forall\, W \subset X, \ \forall\, s \in \mathcal{F}(W)$

$f^{-1}(s(W))$ is open in U.

Now: $f^{-1}(s(W)) = \{ x \in U \mid f(x) \in s(W) \} = \{ x \in U \mid f(x) = s(x) \}$

$f(x) \in \mathcal{F}_x \qquad s(x) \in \mathcal{F}_x$





So $f^{-1}(s(W)) = \{x \in U \mid f(x) = s(x)\}$

is open iff it contains a neighborhood of all of its points,

i.e., iff $\forall x \in f^{-1}(s(W))$ $\exists$ open set $\underset{x}{\overset{}{U_x}} \ni x$, $U_x \subset U$

s.t. $U_x \subset f^{-1}(s(W))$

so $\forall y \in U_x \quad f(y) = s(y)$.

So, if f is continuous, then:

$\forall x \in U$, $\exists$ $U_x \subset U$ open, $x \in U_x$, and a section

$t \in \mathcal{F}(U_x)$ s.t. $\forall y \in U_x$, $f(y) = t(y)$.

Conversely, we can show (using the analysis above) that if f satisfies this condition, then f is continuous.