Definition: The morphism of presheaves

$$\mathcal{F} \to \tilde{\mathcal{F}}$$

is defined by sending $s \in \mathcal{F}(U)$ to the section $x \mapsto s(x)$ where $s(x)$ is the germ of s at x.

Verification of the universal property:

Let $\mathcal{G}$ be a sheaf with a morphism of pre-sheaves $\varphi : \mathcal{F} \to \mathcal{G}$.

Uniqueness of $\tilde{\varphi} : \tilde{\mathcal{F}} \to \mathcal{G}$ s.t.

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\approx"] & \mathcal{G} \\ \tilde{\mathcal{F}} \arrow[ur, "\tilde{\varphi}"'] & \end{tikzcd}$$

Given $U \subset X$ and $f \in \tilde{\mathcal{F}}(U)$,

$\exists$ an open covering $U = \bigcup_{i \in I} W_i$ and sections $s_i \in \mathcal{F}(W_i)$

s.t. for all $x \in W_i$, $f(x) = s_i(x)$.

Then, $\forall i$ $\tilde{\varphi}(s_i) = \varphi(s_i)$ is uniquely determined.

This uniquely determines $\tilde{\varphi}(f)$ because

$\tilde{\varphi}(f)\big|_{W_i} = \tilde{\varphi}(f|_{W_i}) = \tilde{\varphi}(s_i)$ is uniquely determined and $\mathcal{G}$ is a sheaf, so sections of $\mathcal{G}(U)$ are uniquely determined by their restrictions to an open covering.

<u>Existence of $\tilde{\varphi} : \tilde{\mathcal{F}} \to \mathcal{G}$:</u> $U \subset X$ open, $f \in \tilde{\mathcal{F}}(U)$.

$\exists$ covering $U = \bigcup_{i \in I} W_i$ and $s_i \in \mathcal{F}(W_i)$ s.t.

$\forall i$, $\forall x \in W_i$ $f(x) = s_i(x)$, i.e., $f|_{W_i} = s_i$.

So we know $\tilde{\varphi}(f)\big|_{W_i}$. We need to glue these

$$\tilde{\varphi}(f|_{W_i}) = \tilde{\varphi}(s_i) = \varphi(s_i)$$

to get $\tilde{\varphi}(f)$.

We need to know: $\tilde{\varphi}(f|_{W_i})\big|_{W_i \cap W_j} = \tilde{\varphi}(f|_{W_j})\big|_{W_i \cap W_j}$ $\forall\, i,j$

so we need to know: $\varphi(s_i)\big|_{W_i \cap W_j} = \varphi(s_j)\big|_{W_i \cap W_j}$

We know: $\forall\, x \in W_i$ $\quad f(x) = s_i(x)$

So for $x \in W_i \cap W_j$ $\quad f(x) = s_i(x) = s_j(x)$

So $\exists\ W_x \subset W_i \cap W_j$ s.t. $s_i\big|_{W_x} = s_j\big|_{W_x}$

$$W_x \ni x$$

$$\Longrightarrow \varphi(s_i|_{W_x}) = \varphi(s_j|_{W_x})$$

$$\Rightarrow \quad \varphi(s_i)\big|_{W_x} = \varphi(s_j)\big|_{W_x}$$

The W_x form an open covering of $W_i \cap W_j$ and $\mathcal{G}$ is a sheaf, we conclude $\varphi(s_i)\big|_{W_i \cap W_j} = \varphi(s_j)\big|_{W_i \cap W_j}$

Since $\mathcal{G}$ is a sheaf, $\exists\ t \in \mathcal{G}(V)$ s.t. $\forall i$

$$t\big|_{W_i} = \varphi(s_i)$$

We define $\tilde{\varphi}(f) := t \in \mathcal{G}(V)$

We need to verify that this is independent of the choice of open covering $\{W_i\}$ and sections $s_i \in \mathcal{F}(W_i)$ s.t. $f(x) = s_i(x)$ for $x \in W_i$.

__Definitions:__ (1) __Sheaf of rings:__ $\mathcal{F}$ on X sheaf of sets s.t. $\forall\ U$ $\mathcal{F}(U)$ is a ring (commutative with 1) and $\forall\ V \subset U$ the restriction $\mathcal{F}(U) \to \mathcal{F}(V)$ is a hom. of rings.

(2) __Sheaf of modules over a sheaf of rings:__

Given $\mathcal{O}_X$ a sheaf of rings on X, a sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules is a sheaf of abelian groups s.t. $\forall\ U \subset X$ $\mathcal{M}(U)$ is a module over $\mathcal{O}_X(U)$.

s.t. $\forall\ V \subset U$ and all $a \in \mathcal{O}_X(U), m \in \mathcal{M}(U)$

$$(a \cdot m)\big|_V = a\big|_V \cdot m\big|_V$$

(3) <u>Ringed space:</u> $(X, \mathcal{O}_X)$

top. space → X; sheaf of rings → $\mathcal{O}_X$.

(4) <u>Locally ringed space:</u> $(X, \mathcal{O}_X)$ ringed space

s.t. $\forall\ x \in X$, the stalk $\mathcal{O}_{X,x}$ at x is a local ring, i.e., it has exactly one maximal ideal.

(5) <u>Push-forward of a presheaf or sheaf:</u>

Given a continuous map $\varphi : Y \longrightarrow X$ and a presheaf $\mathcal{G}$ on Y, the push-forward $\varphi_* \mathcal{G}$ is the presheaf: $(\varphi_* \mathcal{G})$ on X defined by:

$$\forall\ U \subset X \quad (\varphi_* \mathcal{G})(U) := \mathcal{G}(\varphi^{-1}(U))$$

Exercise: If $\mathcal{G}$ is a sheaf, $\varphi_* \mathcal{G}$ is also a sheaf.

Example: R ring, $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ is a ringed space. We will see that this is in fact a locally ringed space.

(6) Morphism of ringed spaces:

Given $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ ringed spaces.

A morphism of ringed spaces $\varphi : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is the data of a continuous map $\varphi : Y \longrightarrow X$ and a morphism of sheaves of rings

$$\varphi^{\#} : \mathcal{O}_X \longrightarrow \varphi_* \mathcal{O}_Y \quad \text{on } X$$

(7) Morphism of locally ringed spaces: later

Back to $\operatorname{Spec} R$, $\mathcal{O}_{\operatorname{Spec} R}$

Recall that for $f \in R$, $U_f := \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$

$$\mathcal{O}(U_f) = R[f^{-1}] := R[x] \big/ (fx-1)$$

More generally, if we have an integral domain R.

$R \subset \operatorname{Frac} R$ and for any subset $S \subset R$,

we can consider $R[S^{-1}] \subset \operatorname{Frac} R$.

"subring of $\operatorname{Frac} R$ generated by R and the inverses of the elements of S."

We are inspired by the construction of $\mathbb{Q}$ from $\mathbb{Z}$:

$$\mathbb{Q} := \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \big/ \sim \qquad \underset{\rightsquigarrow \frac{a}{b}}{(a,b)} \sim \underset{\rightsquigarrow \frac{c}{d}}{(c,d)} \iff ad - bc = 0$$

$\frac{a}{b}$ = equivalence class of (a,b)

$$\frac{a}{b} = \frac{c}{d} \iff ad - bc = 0.$$

<u>Definition:</u> Let A be a ring. A subset $S \subset A$ is called multiplicative if

(i) $1 \in S$, and

(ii) $\forall\ s,t \in S$, $st \in S$.

<u>Definition:</u> Given $S \subset A$ multiplicative, define an equivalence relation on $A \times S$ as follows:

$$(a,s) \equiv (b,t) \iff \exists\ s' \in S \text{ s.t. } (at-bs)s' = 0.$$

The localization $S^{-1}A$ of A at S is, by definition

$$S^{-1}A := A \times S / \equiv .$$

We usually denote the equivalence class of (a, s) by $\frac{a}{s}$.

Next time: examples.