Remark: $S \subset A$ multiplicative subset

Assume $s \in S$ and $\exists\ b \neq 0$ s.t. $bs = 0$.

$b \mapsto \frac{b}{1} \in S^{-1}A$ $\quad bs = 0 \iff \frac{b}{1} = \frac{0}{1}$

because $\frac{b}{1} = \frac{0}{1} \iff \exists\ s' \in S$ s.t.

$s'(b \cdot 1 - 0 \cdot 1) = 0$

$s'b = 0$

take $s' = s$

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Localizations have a universal property:

Prop.: Let $S \subset A$ be a multiplicative subset.
Let $\varphi : A \to B$ be a ring homomorphism s.t.
$\varphi(s)$ is invertible in $B$, $\forall\ s \in S$. Then there exists a unique homomorphism $S^{-1}\varphi : S^{-1}A \to B$ s.t.

$$S^{-1}\varphi\left(\frac{a}{s}\right) = \varphi(a)\,\varphi(s)^{-1} \quad \forall\ a\in A,\ s\in S.$$

In other words the diagram

$$\begin{tikzcd}
a \arrow[d, mapsto] & A \arrow[r, "\varphi"] \arrow[d] \arrow[r, phantom, shift right=3ex, "\circlearrowleft"] & B \\
\frac{a}{1} & S^{-1}A \arrow[ur, "\exists!\, S^{-1}\varphi"'] &
\end{tikzcd}$$

commutes.

See Atiyah-MacDonald, Chapter 3.

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Our main examples for multiplicative sets are

(1) $S = \{1, f, f^2, f^3, \dots\}$ $\quad f \in A$

(2) $S = A \setminus \mathfrak{p}$ with $\mathfrak{p} \subset A$ prime.

For case (1): $S^{-1}A = A[f^{-1}]$

In case (2): we write $A_{\mathfrak{p}} := S^{-1}A$

Lemma: $A_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}A_\mathfrak{p}$ (the ideal generated by the image of $\mathfrak{p}$ in $A_\mathfrak{p}$).

Proof: Fact: a ring $R$ is local with maximal ideal $\mathfrak{m} \iff \forall x \in R$, $x$ is invertible iff $x \notin \mathfrak{m}$.

$\mathfrak{p}A_\mathfrak{p}$ has this property.

Proposition: The spectrum of a ring is a locally ringed space. In fact, $\forall \mathfrak{p} \in \operatorname{Spec} R$, $\mathcal{O}_\mathfrak{p} = R_\mathfrak{p}$

$\mathcal{O}_\mathfrak{p}$ = the stalk of $\mathcal{O}_{\operatorname{Spec} R}$ at $\mathfrak{p}$

Proof: By def: $\mathcal{O}_\mathfrak{p} = \varinjlim_{\mathfrak{p} \in U} \mathcal{O}(U)$

The proposition follows from the following lemma:

**Lemma:** $$\mathcal{O}_p = \varinjlim_{p \in U_f} \mathcal{G}(U_f) = R_p$$ (natural isomorphisms)

**Proof:** We first prove the first equality.

$$\mathcal{O}_p = \coprod_{p \in U} \mathcal{G}(U) \Big/ \sim \qquad \varinjlim_{p \in U_f} \mathcal{G}(U_f) = \coprod \mathcal{G}(U_f) \Big/ \sim$$

We have the natural inclusion

$$\begin{tikzcd}
\coprod_{p \in U_f} \mathcal{G}(U_f) \arrow[r, hook] \arrow[dr] & \coprod_{p \in U} \mathcal{G}(U) \arrow[d] \\
 & \varinjlim_{p \in U} \mathcal{G}(U) = \mathcal{O}_p
\end{tikzcd}$$

Show that the composition is surjective and its kernel is the equivalence class of $(U_f, 0)$ for any $f \notin p$.

Kernel first: If $(U_f, s) \mapsto 0 \in \varinjlim_{p \in U} \mathcal{O}(U)$,

$$\left( s \in \mathcal{O}(U_f) = R[f^{-1}] \right)$$

then $\exists\ U \subset U_f$ s.t. $s|_U = 0$

Since $U_f$ form a basis of the topology of Spec $R$,

$\exists\ g \in R$ s.t. $p \in U_g \subset U \Rightarrow s|_{U_g} = 0$

$$\Rightarrow (U_f, s) \sim (U_g, 0)$$

Surjectivity of $\varinjlim_{f \notin p} \mathcal{O}(U_f) \to \mathcal{O}_p = \varinjlim_{p \in U} \mathcal{O}(U)$:

Let $(U, s)$ represent an element of $\mathcal{O}_p$. Then, as above, $\exists\ f$ s.t. $p \in U_f \subset U$ $(\Rightarrow f \notin p)$

Then $(U,s) \sim (U_f, s|_{U_f})$, so $s(\mathfrak{p}) := [(U,s)]$
is the image of $[(U_f, s|_{U_f})] = s|_{U_f}(\mathfrak{p})$.
and $\varinjlim_{\mathfrak{p}\in U_f} \mathcal{O}(U_f) \twoheadrightarrow \varinjlim_{\mathfrak{p}\in U} \mathcal{O}(U)$.

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The second equality: $\varinjlim_{f\notin\mathfrak{p}} \mathcal{O}(U_f) = \varinjlim_{f\notin\mathfrak{p}} R[f^{-1}] = R_{\mathfrak{p}}$

For any $f \notin \mathfrak{p}$, we have a natural map, obtained from the universal property of localization:

$$R[f^{-1}] \longrightarrow R_{\mathfrak{p}}.$$

$$\frac{a}{f^n} \longmapsto \frac{a}{f^n}$$

This map induces $\coprod_{f \notin \mathfrak{p}} R[f^{-1}] \longrightarrow R_{\mathfrak{p}}$.

to show that it induces an isomorphism

$$\varinjlim_{f \notin \mathfrak{p}} R[f^{-1}] \xrightarrow{\cong} R_{\mathfrak{p}}$$

we show that its kernel is the equivalence class of $\frac{0}{1}$ and that it is surjective.

Suppose $\frac{a}{f^n}$ maps to $\frac{0}{1} \in R_{\mathfrak{p}}$. By the def. of localization, this means $\exists\, s \in R \setminus \mathfrak{p}$ s.t.

$s(a \cdot 1 - 0 \cdot f^n) = 0$ or $a \cdot s = 0$ $\quad \frac{a}{1} \mapsto \frac{0}{1} \in R[s^{-1}]$

What about $\frac{a}{f^n}$? $\quad \frac{a}{f^n} \mapsto \frac{0}{1}$ in $R[(sf)^{-1}]$ $(asf = 0)$

Injectivity: $\frac{a}{f} \in R_{\mathfrak{p}}$  $f \notin \mathfrak{p}$

then $\frac{a}{f}$ is the image of $\frac{a}{f} \in R[f^{-1}]$ $\square$

More generally, for any $R$-module $M$, we define an $\mathcal{O}$-module $\mathcal{M}$ on $\operatorname{Spec} R$ by setting

$$\mathcal{M}(U_f) := M[f^{-1}] = \text{localization at } f = M \times S/\sim$$

In an analogous way, we have

$$\mathcal{M}(U) = \varprojlim_{U_f \subset U} M[f^{-1}]$$

and for all $\mathfrak{p} \in \operatorname{Spec} R$ $\mathcal{M}_{\mathfrak{p}} = M_{\mathfrak{p}} = \varinjlim_{f \notin \mathfrak{p}} M[f^{-1}]$

Def: A sheaf of $\mathcal{O}_R$-modules is called quasi-coherent if it can be obtained from an $R$-module in the manner above.

Def: An affine scheme is a locally ringed space which is isomorphic, as a locally ringed space, to the spectrum of a ring $A$. A scheme is a locally ringed space (or just a ringed space) which has a covering by open sets that are affine schemes.

Def: Morphisms of locally ringed spaces:

(1) Given two local rings $(A, m)$ $(B, n)$, a homomorphism of rings $\varphi: A \to B$ is called local

if $\quad \varphi^{-1} n = m$.

(2) A morphism of locally ringed spaces

$$\varphi : (Y, \mathcal{O}_Y) \longrightarrow (X, \mathcal{O}_X) \qquad Y \xrightarrow{\varphi} X \quad \varphi^{\#} : \mathcal{O}_X \longrightarrow \varphi_* \mathcal{O}_Y$$

is a morphism of ringed spaces s.t. $\forall\, q \in Y$

the induced morphism
$$\begin{tikzcd} \mathcal{O}_{X,\varphi(q)} \arrow[r] \arrow[rr, bend right] & (\varphi_* \mathcal{O}_Y)_{\varphi(q)} \arrow[r] & \mathcal{O}_{Y,q} \end{tikzcd}$$

$$\left( \begin{array}{ccc} (\varphi_* \mathcal{O}_Y)_{\varphi(q)} & \longrightarrow & \mathcal{O}_{Y,q} \\ \| & & \| \\ \varinjlim\limits_{\substack{q \in \varphi^{-1}(U) \\ \text{or } \varphi(q) \in U}} \mathcal{O}_Y(\varphi^{-1}(U)) & & \varinjlim\limits_{q \in V} \mathcal{O}_Y(V) \end{array} \right)$$

is a local homomorphism of local rings.