__Def__: In general, for any kind of structure, an isomorphism is a morphism which has a two-sided inverse.

__First example__: $\operatorname{Spec} R$ is a locally ringed space.

Affine spaces: $\mathbb{A}^n_R := \operatorname{Spec} R[x_1, \dots, x_n]$.

__Remark__: The points of $\operatorname{Spec} R$ are not always closed, i.e., given $\underline{\mathfrak{p} \subset R}$ prime $\{\mathfrak{p}\} \subset \operatorname{Spec} R$

$$\{\mathfrak{p}\} \subset \{\mathfrak{p}\} \subset \operatorname{Spec} R$$

$\overline{\{\mathfrak{p}\}}$ = smallest closed set $V(I)$ containing $\mathfrak{p}$

= smallest $V(I)$ s.t. $I \subset \mathfrak{p}$

= $V(I)$ with I largest ideal contained in $\mathfrak{p}$.

$\Rightarrow \quad \overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \supset \mathfrak{p}\}$

So $\{\mathfrak{p}\}$ is closed $\Leftrightarrow$ $\{\mathfrak{p}\} = \overline{\{\mathfrak{p}\}}$

$\Leftrightarrow \quad \{\mathfrak{q} \mid \mathfrak{q} \supset \mathfrak{p}\} = \{\mathfrak{p}\}$

$\Leftrightarrow \quad \mathfrak{p}$ is maximal.

Conclusion: Lemma: The closed points of $\operatorname{Spec} R$ are the maximal ideals.

Corollary: The closed points of $\mathbb{A}^n_k$ can be identified with k^n when k is algebraically closed.

Before talking about projective spaces, we need a few more tools.

Back to general sheaves and schemes:

We know that any morphism of (pre)sheaves induces morphisms on all the stalks.

Lemma: A morphism of sheaves is an isomorphism if and only if it induces isomorphisms on all the stalks.

Lemma: Given two rings A, B, the datum of a morphism $\operatorname{Spec} A \longrightarrow \operatorname{Spec} B$ is equivalent to the datum of a homomorphism of rings $B \longrightarrow A$.

We will in fact prove the stronger:

Lemma: Suppose given a scheme X and a ring A. The natural map

$$\mathrm{Hom}(X, \operatorname{Spec} A) \longrightarrow \mathrm{Hom}(A, \mathcal{O}_X(X))$$

obtained by sending $\varphi : X \to \operatorname{Spec} A$ to its global sections

$$\begin{array}{ccc} \varphi^\#(\operatorname{Spec} A) : \mathcal{O}_A(\operatorname{Spec} A) & \longrightarrow & \varphi_* \mathcal{O}_X(\operatorname{Spec} A) \\ \| & & \| \\ A & \longrightarrow & \mathcal{O}_X(X) \end{array}$$

is a bijection.

Proof: Suppose given a morphism of rings $\alpha : A \to \mathcal{O}_X(X)$.

We first construct a morphism $\varphi : X \to \operatorname{Spec} A$ inducing α.

To define the map of sets $X \to \operatorname{Spec} A$, let x be a point of X, what should the image of x in $\operatorname{Spec} A$ be?

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[dr, "\gamma_x \circ \alpha"'] & \mathcal{O}_X(X) \arrow[d, "\gamma_x"] & \ni s \arrow[d, mapsto] \\
 & \mathcal{O}_{X,x} & \ni s(x)
\end{tikzcd}$$

local ring, let m_x be its maximal ideal.

Define $\varphi(x) := (\gamma_x \circ \alpha)^{-1}(m_x)$.

Need to show: $\varphi : X \to \operatorname{Spec} A$ is continuous.

We show that $\forall f \in A$, $\varphi^{-1}(U_f) \subset X$ is open.

$$U_f = \{\mathfrak{p} \subset A \mid f \notin \mathfrak{p}\}$$

$$\begin{aligned}
\varphi^{-1}(U_f) &= \{x \in X \mid (\gamma_x \circ \alpha)^{-1}(\mathfrak{m}_x) \not\ni f\} \\
&= \{x \in X \mid \gamma_x \circ \alpha(f) \notin \mathfrak{m}_x\} \\
&= \{x \in X \mid \gamma_x \circ \alpha(f) \in \mathcal{O}_{X,x} \text{ is invertible}\} \\
&= \{x \in X \mid \exists\, h_x \in \mathcal{O}_{X,x} \text{ s.t. } h_x \cdot (\gamma_x \circ \alpha)(f) = 1\} \\
&= \{x \in X \mid \exists\, U_x \text{ open neighborhood of } x \\
&\qquad \text{and } \exists\, h \in \mathcal{O}_X(U_x) \text{ s.t.} \\
&\qquad h(x) \cdot (\gamma_x \circ \alpha)(f) = 1 \}
\end{aligned}$$

$$\begin{tikzcd}
f \in A \arrow[r, "\alpha"] & \mathcal{O}_X(X) \arrow[d, "\gamma_x"'] \arrow[dr, "|_U"] & \\
& \mathcal{O}_{X,x} & \mathcal{O}_X(U) \arrow[l, "\gamma_{U,x}"]
\end{tikzcd}$$

$x \in U$

$(\gamma_x \circ \alpha)(f) =$ germ of $\alpha(f)|_U$ at x

$\forall\, U \ni x$

We have $h(x)\cdot\underbrace{(\alpha(f))(x)}_{\overset{\shortparallel}{(\gamma_x\circ\alpha)(f)}} = 1$

$\Rightarrow \exists \ \underset{\underset{x}{\cup}}{V_x} \subset U_x$ s.t. $h\big|_{V_x}\cdot \alpha(f)\big|_{V_x} = 1$

So, for every $y \in V_x$ $h(y)\cdot(\alpha(f))(y) = 1$.

$\Rightarrow V_x \subset \varphi^{-1}(U_f)$

and $\varphi^{-1}(U_f)$ is a neighborhood of x $\forall x \in \varphi^{-1}(U_f)$

Now we define the homomorphism of sheaves

$$\varphi^{\#}: \mathcal{O}_A \longrightarrow \varphi_*\mathcal{O}_X \quad \text{on } \operatorname{Spec} A$$

We first do it on basic open sets.

Let $f \in A$

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] \arrow[dr, "\alpha|_{U_f}"] & \mathcal{O}_X(X) = \varphi_* \mathcal{O}_X(\operatorname{Spec} A) \arrow[d, "\text{restriction}"] \\ A[f^{-1}] \arrow[r, dashed, "?"] & \varphi_* \mathcal{O}_X(U_f) = \mathcal{O}_X(\varphi^{-1}(U_f)) \end{tikzcd}$$

$$\varphi^{-1}(U_f) = \{ x \in X \mid \text{germ of } \alpha(f) \text{ at } x \text{ is invertible} \}$$
$$\overset{\text{def.}}{=} X_{\alpha(f)}$$

To show $\alpha|_{U_f}$ factors through $A[f^{-1}]$, we need to show that $\alpha(f)|_{U_f}$ is invertible in $\mathcal{O}_X(\varphi^{-1}(U_f))$.

We know that $\alpha(f)(x)$ is invertible $\forall x \in \varphi^{-1}(U_f)$.

So there is an open covering $\varphi^{-1}(U_f) = \bigcup_{i \in I} V_i$ and sections $h_i \in \mathcal{O}_X(V_i)$ s.t. $f|_{V_i} \cdot h_i = 1$. $\forall i$

Claim: the h_i glue together to a section $h \in \mathcal{O}_X(\varphi^{-1}(U_f))$ s.t. $h \cdot \alpha(f)|_{\varphi^{-1}(U_f)} = 1$

proof: exercise $\left(V_i \cap V_j \quad h_i|_{V_i \cap V_j} \cdot \alpha(f)|_{V_i \cap V_j} = 1 \right)$

So $\alpha(f)|_{\varphi^{-1}(U_f)}$ is invertible and we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}_X(X) \arrow[d] \\ A[f^{-1}] \arrow[r, "\alpha_f"] & \mathcal{O}_X(\varphi^{-1}(U_f)) \end{tikzcd}$$

We need to show that $\forall\ f, g \in A$ s.t. $U_g \subset U_f$, we have a commutative diagram

$$\begin{tikzcd}
h & A \arrow[r, "\alpha"] \arrow[d] & \mathcal{O}_X(X) \arrow[d, "|"] & \alpha(h) \\
\frac{h}{f^m} & A[f^{-1}] \arrow[r, "\alpha_f"] \arrow[d] \arrow[dr] & \mathcal{O}_X(\varphi^{-1}(U_f)) \arrow[d, "|"] & \alpha(h)\alpha(f)^{-m} \\
A[f^{-1}][(\tfrac{g}{1})^{-1}] = & A[g^{-1}] \arrow[r, "\alpha_g"] & \mathcal{O}_X(\varphi^{-1}(U_g)) &
\end{tikzcd}$$

Recall: what it means for U_f to contain U_g:

$$U_g \subset U_f$$

$$\{p \not\ni g\} \subset \{p \not\ni f\}$$

$$\{p \ni g\} \supset \{p \ni f\}$$

$$\sqrt{(f)} = \bigcap_{f \in p} p \ni g \qquad \Leftrightarrow \exists\, n>0,\ \exists\, l \in A \text{ s.t. } g^n = fl$$