Now glue for arbitrary open sets (or pass to the inverse limit) to obtain the homomorphism of sheaves $\varphi^{\#}: \mathcal{O}_{\operatorname{Spec} A} \longrightarrow \varphi_* \mathcal{O}_X$. $\varphi_x : \mathcal{O}_{A,x} \to \mathcal{O}_{X,x}$

Also note that $\varphi_x^{-1}(\mathfrak{m}_x) = \mathfrak{p} A_{\mathfrak{p}} \quad \forall\, x \in X$, so we indeed have a morphism of locally ringed spaces.

<u>Uniqueness or injectivity:</u> If two morphisms $\varphi, \psi : X \longrightarrow \operatorname{Spec} A$ induce the same map on global sections, then $\forall\, x \in X$, we have the commutative diagram

$\varphi^{\#}, \psi^{\#} : A \longrightarrow \mathcal{O}_X(X)$, and $\varphi_x^{-1}(\mathfrak{m}_x) = \mathfrak{p} A_{\mathfrak{p}}$
$\psi_x^{-1}(\mathfrak{m}_x) = \mathfrak{q} A_{\mathfrak{q}}$.

$$\begin{tikzcd} & A \arrow[dl, "\gamma_{\mathfrak{p}}"'] \arrow[d, "\gamma_{\mathfrak{q}}"] \arrow[r] & \mathcal{O}_X(X) \arrow[d, "\gamma_x"] \\ A_{\mathfrak{p}} \arrow[rr, bend right, "\varphi_x"'] & A_{\mathfrak{q}} \arrow[r, "\psi_x"] & \mathcal{O}_{X,x} \end{tikzcd}$$

$\mathfrak{p} := \varphi(x)$, $\mathfrak{q} = \psi(x)$

pull back to A:

$\mathfrak{p} = \gamma_{\mathfrak{p}}^{-1}(\mathfrak{p} A_{\mathfrak{p}}) = (\varphi^{\#})^{-1}(\gamma_x^{-1}(\mathfrak{m}_x)) = (\psi^{\#})^{-1}(\gamma_x^{-1}(\mathfrak{m}_x)) = \gamma_{\mathfrak{q}}^{-1}(\mathfrak{q} A_{\mathfrak{q}}) = \mathfrak{q}$

So φ and ψ agree on topological spaces.
The maps $\varphi^{\#}$ and $\psi^{\#}$ are also equal because they are both obtained by localizing the map on the global sections. $\square$

<u>Generalization of affine varieties</u>: $Y \subset k^n$
$= V(I)$
coordinate ring of Y: $k[x_1, \dots, x_n]/I(Y)$ $\quad I \subset k[x_1, \dots, x_n]$.
$\longleftrightarrow$ scheme $\operatorname{Spec} k[x_1, \dots, x_n]/I(Y)$.

How do we generalize projective space and projective varieties?

The Proj of a graded ring:

<u>Recall</u>: A graded ring S (see Atiyah-McDonald) is a commutative ring with 1 and a direct sum decomposition $S = \bigoplus_{\substack{d \in \mathbb{Z} \\ d \geq 0}} S_d$ of abelian groups.

s.t. $S_d \cdot S_e \subset S_{d+e}$

<u>Note</u>: $S_0 \subset S$ is a subring.

<u>Notation</u>: $S_+ := \bigoplus_{d>0} S_d \subset S$ is an ideal

Definition: As a set

$$\operatorname{Proj} S := \{ \mathfrak{p} \mid \mathfrak{p} \subset S \text{ homogeneous prime}, \ \mathfrak{p} \not\supset S_+ \}.$$

The closed sets of the topology on $\operatorname{Proj} S$ are:

$$Z(I) := \{ \mathfrak{p} \mid \mathfrak{p} \supset I \} \subset \operatorname{Proj} S$$

for all $I \subset S$ homogeneous ideals.

We have basic open sets: $\forall \ f \in S$ homogeneous

$$U_f := \{ \mathfrak{p} \mid \mathfrak{p} \not\ni f \} \subset \operatorname{Proj} S.$$

These form a basis of the topology, as in the affine case.

For a basic open set U_f, we define the ring of U_f :

$\mathcal{O}_{\operatorname{Proj} S}(V_f) := S[f^{-1}]_0$ the elements of degree 0 in the localization.

("dehomogenization".

think $k\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$.

$= \left(k[x_0, \dots, x_n][x_i^{-1}]\right)_0$)

<u>Lemma:</u> $\forall\, \mathfrak{p} \in \operatorname{Proj} S$, we have a canonical isomorphism $\mathcal{O}_{\operatorname{Proj} S, \mathfrak{p}} \cong S_{\mathfrak{p}, 0}$ the subring of elements of degree 0.

<u>Proof:</u> Taking direct limits commutes with taking elements of degree 0. $\square$

In particular, $\operatorname{Proj} S$ is a locally ringed space.

<u>Lemma:</u> For all homogeneous elements $f \in S$, we have
$$U_f \cong \operatorname{Spec} S[f^{-1}]_0$$
as locally ringed spaces. We endow U_f with the topology induced from $\operatorname{Proj} S$ and the sheaf of rings: $\mathcal{O}_{U_f}(V) = \mathcal{O}_{\operatorname{Proj} S}(V) \quad \forall\, V \subset U$.

<u>Proof:</u> Localization commutes with taking subrings of elements of degree 0. First verify that the topological spaces can be identified:
$$\underset{\text{hom. prime}}{f \notin \mathfrak{p} \subset S} \quad \Longleftrightarrow \quad \underset{\text{prime}}{\mathfrak{p}_0 \subset S[f^{-1}]_0}$$

$f \notin \mathfrak{p} \subset S$ homogeneous prime

$\iff \quad \underbrace{\mathfrak{p} S[f^{-1}]}_{\text{ideal generated by the image of } \mathfrak{p}} \subset S[f^{-1}]$ homogeneous prime

$\rightsquigarrow \quad (\mathfrak{p} S[f^{-1}]) \cap S[f^{-1}]_0 \subset S[f^{-1}]_0$

prime ideal

get a map $\quad U_f \longrightarrow \operatorname{Spec} S[f^{-1}]_0$

Now start with a prime ideal $\mathfrak{q}_0 \subset S[f^{-1}]_0$

want to construct a homogeneous ideal

$\mathfrak{q} \subset S[f^{-1}]$ s.t. $\quad \mathfrak{q} \cap S[f^{-1}]_0 = \mathfrak{q}_0$

(homogenization) $\qquad \mathfrak{q}_0 S[f^{-1}] \subset S[f^{-1}]$

page 149 of "The rising sea" by Ravi Vakil $\qquad$ ideal generated by $\mathfrak{q}_0$.