<u>Definition</u>: Given a ring R, we define projective n-space over R: $\mathbb{P}^n_R := \operatorname{Proj} R[x_0, \dots, x_n]$

<u>First properties of schemes</u>:

<u>Def</u>: (1) A scheme is connected/irreducible/quasi-compact if its underlying topological space is connected/irreducible/quasi-compact.

(2) A scheme is reduced if, for all open sets $U \subset X$ $\mathcal{O}_X(U)$ is reduced, i.e., has no nilpotent elements. Equivalently (homework), for all points $x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced.

(3) A scheme is integral if, for all open sets $U \subset X$,

the ring $\mathcal{O}_X(U)$ is an integral domain $(=0 \Leftrightarrow U=\phi)$

(4) A scheme is locally Noetherian if it can be covered by affine sets Spec$A$ with A noetherian.

(5) A scheme is Noetherian if it is locally Noetherian and quasi-compact. Equivalently, it has a <u>finite</u> cover by open affine sets with noetherian rings of sections.

<u>Prop.</u>: A scheme is integral if and only if it is irreducible and reduced.

<u>Proof</u>: Clearly, an integral scheme is reduced. Suppose X is integral.

If $X = X_1 \cup X_2$ with X_1, X_2 closed.

Put $U_i = X \setminus X_i$: this is open.

$$U_1 \cap U_2 = (X \setminus X_1) \cap (X \setminus X_2) = X \setminus (X_1 \cup X_2)$$
$$= \phi$$

The sheaf property implies

$$\mathcal{O}_X(U_1 \amalg U_2) \xrightarrow{\text{product of restriction maps.}} \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$$
as rings.

$\mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ is <u>not</u> an integral domain:

$$(1,0)\cdot(0,1) = (0,0)$$

$(1,0)$ and $(0,1)$ are zero divisors unless one of them is $(0,0)$, which means that in $\mathcal{O}_X(U_1)$ or $\mathcal{O}_X(U_2)$, we have $1=0$ $\Rightarrow$ $\mathcal{O}_X(U_1) = 0$ or $\mathcal{O}_X(U_2) = 0$

$\Rightarrow \quad V_1 = \emptyset \quad \text{or} \quad V_2 = \emptyset$

$\Rightarrow \quad X_1 = X \quad \text{or} \quad X_2 = X$.

Conversely, suppose X is irreducible and reduced.

Let $U \subset X$ be open, $U \neq \emptyset$.

Choose $f, g \in \mathcal{O}_X(U)$ s.t. $fg = 0$.

Recall $Z(f) := \{x \in U \mid f(x) \in m_x\}$ is closed in U

$Z(g) := \{x \in U \mid g(x) \in m_x\}$ "

Recall (homework) that a non-empty open subset of an irreducible space is irreducible.

$fg = 0 \Rightarrow Z(f) \cup Z(g) = Z(fg) = U$

If $U \neq \emptyset$, then either $Z(f) = U$ or $Z(g) = U$.

If $Z(f) = U$, then for any open affine $\operatorname{Spec} A \subset U$, we have $f(\mathfrak{p}) \in \mathfrak{p} A_{\mathfrak{p}} \quad \forall\, \mathfrak{p} \subset A$ prime

or $f|_{\operatorname{Spec} A} \in \mathfrak{p} \quad \forall\, \mathfrak{p} \subset A$ prime

$$\Rightarrow \quad f|_{\operatorname{Spec} A} \in \text{nilradical} = \bigcap_{\mathfrak{p} \subset A \text{ prime}} \mathfrak{p}$$

$$= \{\text{nilpotent elements}\} \subset A$$

$\Rightarrow$ $f|_{\operatorname{Spec} A}$ is nilpotent.

$\Rightarrow$ $f|_{\operatorname{Spec} A} = 0$ because A is reduced.

U is covered by open affine subsets, sheaf property

$\Rightarrow$ $f = 0$. $\square$.

<u>Proposition</u>: A scheme is locally noetherian if and only if, for every open affine $U = \operatorname{Spec} A \subset X$, A is noetherian.

In particular, an affine scheme $X = \operatorname{Spec} A$ is (locally) noetherian iff A is noetherian.

Proof: The "if" part is the definition.

For the other direction, assume X is locally noetherian, let $U = \operatorname{Spec} A \subset X$ be open affine.

We will show that A is noetherian.

X has a cover by open affine sets with Noetherian rings. If B is noetherian, then by the Hilbert basis theorem, $B[f^{-1}]$ is noetherian $\forall\, f \in B$.

$$= B[x]/(fx-1)$$

$\Rightarrow$ open sets of the form $\operatorname{Spec} C$ with C noetherian form a basis of the topology of X.