$U = \operatorname{Spec} A$ $\quad$ U has a covering by open sets of the form $V = \operatorname{Spec} B$ with B noetherian.

$$V = \operatorname{Spec} B \subset U = \operatorname{Spec} A \qquad A \xrightarrow{\text{restriction}} B$$

$\exists$ $f \in A$ s.t. $U_f \subset \operatorname{Spec} B$

If $\bar{f}$ is the image of f in B, then

$$U_f = V_{\bar{f}} \quad \text{and} \quad A[f^{-1}] = B[\bar{f}^{-1}]$$

$$\operatorname{Spec} A \supset U_f = \{ \mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p} \}$$

$$= \{ \mathfrak{p} \in \operatorname{Spec} A \mid f(\mathfrak{p}) \notin \mathfrak{p} A_{\mathfrak{p}} \} \subset \operatorname{Spec} B.$$

$$= \{ \mathfrak{q} \in \operatorname{Spec} B \mid \bar{f}(\mathfrak{q}) \notin \mathfrak{q} B_{\mathfrak{q}} \}$$

$$= \{ \mathfrak{q} \in \operatorname{Spec} B \mid \bar{f} \notin \mathfrak{q} \} = U_{\bar{f}} \subset \operatorname{Spec} B$$

$\Rightarrow$ the rings are the same because these are the rings of X $\Rightarrow$ $A[f^{-1}] = B[\bar{f}^{-1}]$ is noetherian.

We can cover U with a finite number of open sets $U_{f_1}, \dots, U_{f_r}$ which are affine with noetherian rings. $U = U_{f_1} \cup \dots \cup U_{f_r}$

In particular, $f_1, \dots, f_r$ generate the unit ideal.

Now we show A is noetherian:

Let $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \mathfrak{a}_3 \subset \dots$

be an ascending chain of ideals of A.

For each i, the chain of ideals

$$\mathfrak{a}_1 A[f_i^{-1}] \subset \mathfrak{a}_2 A[f_i^{-1}] \subset \mathfrak{a}_3 A[f_i^{-1}] \subset \cdots$$

is stationary because $A[f_i^{-1}]$ is noetherian.

The original chain is stationary by the following lemma:

<u>Lemma:</u> Let A be a ring, $f_1, \dots, f_n \in A$ which generate the unit ideal. Then, for any ideal $\mathfrak{a} \subset A$,

$$\mathfrak{a} = \bigcap_{i=1}^{n} \varphi_i^{-1}\left(\varphi_i(\mathfrak{a})A[f_i^{-1}]\right)$$

where $\varphi_i : A \to A[f_i^{-1}]$ is the localization map.

<u>Proof:</u> Clearly $\mathfrak{a} \subset \bigcap_{i=1}^{n} \varphi_i^{-1}\left(\varphi_i(a)A[f_i^{-1}]\right)$.

For the reverse inclusion, choose an element

$b \in \bigcap_{i=1}^{n} \varphi_i^{-1}\left(\varphi_i(\mathfrak{a})A[f_i^{-1}]\right)$

For each i, there exists $n_i \in \mathbb{Z}_+$, $a_i \in \mathcal{O}_B$

s.t. $\varphi_i(b) = \dfrac{a_i}{f_i^{n_i}} = \dfrac{f_i^l a_i}{f_i^{n_i+l}}$ (we can replace a_i with $f_i^l a_i \cdots$)

We can increase some of the n_i so that they become equal: $\forall i$ $\varphi_i(b) = \dfrac{a_i}{f_i^n}$

$$\varphi_i(b) = \frac{b}{1}$$

$\exists\, m_i \in \mathbb{Z}_+$ s.t. $f_i^{m_i}(f_i^n b - a_i) = 0$

Again, increase some of the m_i so they become equal:

$\exists\, m, n$ s.t. $f_i^m(f_i^n b - a_i) = 0 \quad \forall i$

$\Rightarrow \forall i \quad f_i^{m+n} b \in \mathcal{O}_B$.

$f_1, \dots, f_r$ generate the unit ideal $\iff U_{f_1} \cup \dots \cup U_{f_r} = \operatorname{Spec} A$

$\forall i \qquad U_{f_i} = U_{f_i^{m+n}}$

So $\quad f_1^{m+n}, \dots, f_r^{m+n}$ generate the unit ideal.

$\Rightarrow \exists \ c_1, \dots, c_r \in A \quad$ s.t. $\quad 1 = \sum_{i=1}^{r} c_i f_i^{m+n}$

$$b = b \cdot 1 = b \sum_{i=1}^{r} c_i f_i^{m+n} = \sum_{i=1}^{r} c_i (f_i^{m+n} b)$$

$$\in \mathcal{B} \qquad \square$$

Properties of morphisms of schemes:

Definition: (1) A morphism of schemes $f: X \longrightarrow Y$ is locally of finite type if $\exists$ a covering of Y by open affine subsets $V_i = \operatorname{Spec} B_i$ s.t. $\forall i$ $f^{-1}(V_i)$ has a covering by open affine subsets $U_{ij} = \operatorname{Spec} A_{ij}$ where $\forall i, j$ A_{ij} is a finitely generated B_i-algebra

$$\left(\text{via} \quad f : X \to Y \qquad f^{\#} : \mathcal{O}_Y \to f_* \mathcal{O}_X \right.$$

$$\left. \begin{tikzcd} B_i = \mathcal{O}_Y(V_i) \arrow[r] \arrow[dr] & \mathcal{O}_X(f^{-1}(V_i)) \arrow[d] \\ & \mathcal{O}_X(U_{ij}) = A_{ij} \end{tikzcd} \right)$$

(2) The above morphism is of finite type if, in addition, each $f^{-1}(V_i)$ can be covered with a finite number of the U_{ij}.

(3) A morphism of schemes $f : X \to Y$ is finite if $\exists$ a covering of Y by open affine sets $V_i = \operatorname{Spec} B_i$ s.t. $\forall i$ $f^{-1}(V_i)$ is affine $= \operatorname{Spec} A_i$ with A_i a finite B_i-algebra (i.e., A_i is a finitely generated B_i-module)

$$\left(B_i = \mathcal{O}_Y(V_i) \longrightarrow \mathcal{O}_X(f^{-1}(V_i)) = A_i \right)$$

(4) An open subscheme of a scheme X is an open subset U of X with topology induced from X and sheaf of rings $\mathcal{O}_U := \mathcal{O}_X|_U$, meaning

$$\mathcal{O}_U(V) = \mathcal{O}_X(V) \quad \forall \quad V \underset{\text{open}}{\subset} U.$$

(5) An open embedding is a morphism of schemes $f: X \hookrightarrow Y$ which induces an isomorphism of X with an open subscheme of Y.

(6) A closed embedding is a morphism of schemes $f: X \hookrightarrow Y$ which induces a homeomorphism of X with a closed subset of Y, and such that the morphism of sheaves $f^\#: \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective. (The kernel of $f^\#$ is then a sheaf of ideals on Y)

(7) A closed subscheme X of a scheme Y is an equivalence class of closed embeddings $Z \overset{i}{\hookrightarrow} Y$, where two closed embeddings $i: Z \hookrightarrow Y$, $i': Z' \hookrightarrow Y$ are

equivalent if $\exists$ an isomorphism $\varphi : Z \xrightarrow{\sim} Z'$ s.t.

$$\begin{tikzcd} Z \arrow[rr, "\sim"] \arrow[dr, "i"'] & & Z' \arrow[dl, "i'"] \\ & Y & \end{tikzcd}$$

$$i' \circ \varphi = i .$$

In other words, a closed subscheme is a closed subset $X \hookrightarrow Y$ with a sheaf of rings $\mathcal{O}_X$ and a morphism of schemes $f : X \hookrightarrow Y$ s.t. $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is surjective.