Some examples:

(1) Varieties (affine, quasi-affine, projective, quasi-projective varieties) over a field k are schemes of finite type over k. The definition also includes "separated" which we will see soon.

(2) $A \subset K = \mathrm{Frac}(A)$, A an integral domain

$B :=$ the integral closure of A in K.

$:=$ the set of elements of K which are integral/A, meaning they satisfy a monic polynomial with coefficients in A

$$:= \{ x \in K : \exists\ a_1, \dots, a_n \in A \text{ s.t. } x^n + a_1 x^{n-1} + \dots + a_n = 0 \}$$

(Atiyah-McDonald): B is a subring of K and is a finitely generated A-module.

$A \subset B$ induces a morphism of schemes $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ which is a finite morphism.

$\operatorname{Spec} B$ is the "normalization" of $\operatorname{Spec} A$.

<u>Example</u>: $A := k[x,y]/(x^3-y^2)$ the coordinate ring of a cuspidal cubic

$$= A(Y)$$

$$Y = Z(x^3-y^2) \subset \mathbb{A}^2_k$$

$K := k(t) \hookleftarrow A$ identifies K with $\operatorname{Frac}(A)$.

$t^2 \longleftarrow x$ $\qquad$ t satisfies $X^2 - x = 0$

$t^3 \longleftarrow y$ $\qquad$ and $X^3 - y = 0$

So $k[t] \subset k(t)$ is contained in the integral closure of A in K

One can show that $k[t]$ is integrally closed.

So $k[t]$ is the integral closure of A in K.

(3) X a scheme, $f \in \mathcal{O}_X(X)$

$$X_f \hookrightarrow X \quad \text{is open}$$

$$X_f \hookrightarrow X \quad \text{is an open embedding}$$

(4) A a ring $\qquad \operatorname{Spec} A$

$I \subset A$ an ideal. $A \twoheadrightarrow A/I$ induces

a closed embedding $\operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$.

We think of this as the closed subscheme of $\operatorname{Spec} A$ defined by the ideal I. The morphism of sheaves of rings is surjective because it is surjective on the stalks.

In this way, every ideal, not just radical ones, defines a closed subscheme of $\mathbb{A}^n_k$.

For instance, in $\mathbb{A}^2$, we can look at

$$Z(x) \quad \text{or} \quad Z(x^2) \hookrightarrow \operatorname{Spec} k[x,y]/_{x^2} \hookrightarrow \mathbb{A}^2_k$$

$$\Updownarrow$$

$$\operatorname{Spec} k[x,y]/_{(x)} \hookrightarrow \operatorname{Spec} k[x,y] = \mathbb{A}^2_k$$

Both of these subschemes are "supported" on the y-axis (their underlying topological space).

x is a nilpotent in the ring of $Z(x^2)$.

Schemes with nilpotents naturally occur as "limits" of reduced schemes. E.g.: $Z(x^2-ty^2)$ for different values of $t \in k$. If $t \neq 0$, $Z(x^2-ty^2)$ has ring $k[x,y]/(x^2-ty^2)$ which is reduced.

If $t=0$, $Z(xy)/(x^2)$ has nilpotents.

What the above shows is that, in general, there are many ways of putting a scheme structure on a closed subset of a scheme. The set of scheme structures on $\underset{\text{closed}}{Y \subset X}$ has a minimal element: the reduced induced structure.

The reduced induced structure is determined by the closed subset.

<u>Def</u>: Given a closed subset $Y \subset X$ scheme, the reduced induced scheme structure on Y is defined as follows. For any affine open set $U = \operatorname{Spec} A \hookrightarrow X$, we let the ideal of $Y \cap U$ be the intersection of all the prime ideals of A belonging to $Y \cap U$.

In other words, if $Y = V(I)$ for $I \subset A$ ideal, then, the ideal of the reduced induced scheme structure on Y is $\sqrt{I} = \bigcap_{p \supset I} p$.

To verify that the above definition makes sense, we need to check that on an intersection $U \cap Y$, the scheme structures induced from U and V on Y are equal.

<u>Dimension:</u> The dimension of a scheme is its dimension as a topological space (i.e., the maximum length of a chain of irreducible closed subsets). For any irreducible closed subset Z of X, the codimension of Z in X is the supremum of the set of integers n such that $\exists$ a chain of irreducible closed subsets $Z = Z_0 \subsetneq Z_1 \subsetneq Z_2 \subsetneq \cdots \subsetneq Z_n \subset X$.

Caution: For varieties over a field,

$$\dim Z + \operatorname{codim} Z = \dim X.$$

This is not true in general for schemes.
(see II.3.2.8, Exercises II.3.20, 21, 22)

Example: $\dim \operatorname{Spec} A = \text{Krull} \dim A$.

Fibers of a morphism: can they be schemes?
YES: fiber products.

Fiber products: Def: $X \xrightarrow{\pi_X} S$, $Y \xrightarrow{\pi_Y} S$
morphisms of schemes.

a scheme is called a fiber product of X and Y over S, denoted $X \times_S Y$ if $\exists$ commutative.

$$\begin{tikzcd} X \times_S Y \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\pi_X"] \\ Y \arrow[r, "\pi_Y"'] & S \end{tikzcd}$$

s.t. $\forall$ scheme $Z \xrightarrow{\pi_Z} S$ and morphism $f: Z \to X$, $g: Z \to Y$ s.t. $\pi_X f = \pi_Z = \pi_Y g$, i.e.,

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "\pi_X"] \\ Y \arrow[r, "\pi_Y"'] & S \end{tikzcd}$$

$\exists!$ $h: Z \longrightarrow X \times_S Y$ s.t. $p_1 h = f$, $p_2 h = g$

i.e.,

$$\begin{tikzcd}
Z \arrow[dr, "h"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] \arrow[ddrr, bend right=60, "\pi_Z"'] & & \\
& X \times_S Y \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\pi_X"] \\
& Y \arrow[r, "\pi_Y"'] & S
\end{tikzcd}$$

commutes.

<u>Notation:</u> When we have a fiber product, we write

$$\begin{tikzcd}
X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
Y \arrow[r] & S
\end{tikzcd}$$

__Note:__ The universal property implies that a fiber product, if it exists, is unique up to S-isomorphism

(i.e.,
$$\begin{tikzcd} Z \arrow[rr, "\varphi"] \arrow[dr, "\pi_Z"'] & & Z' \arrow[dl, "\pi_{Z'}"] \\ & S & \end{tikzcd}$$
commutes)

__Existence:__ We first construct fiber products for affine schemes, then glue.

$$S = \operatorname{Spec} R \qquad X = \operatorname{Spec} A \qquad Y = \operatorname{Spec} B.$$

$R \longrightarrow A$, $R \longrightarrow B$ corresponding hom. of rings.

$$\begin{tikzcd} 1\otimes b & A \otimes_R B & A \arrow[l] & a\otimes 1 & a \arrow[l, mapsto] \\ b \arrow[u, mapsto] & B \arrow[u] & R \arrow[l] \arrow[u] & & \end{tikzcd}$$

$$r \longmapsto 1 \otimes r = r \otimes 1$$

define $X \underset{S}{\times} Y := \operatorname{Spec} A \underset{R}{\otimes} B.$

Universal property:

$$\begin{tikzcd} Z \arrow[rr, "f"] \arrow[dr, "\pi_Z"'] & & X \arrow[dl, "\pi_X"] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "\pi_Z"'] & & Y \arrow[dl, "\pi_Y"] \\ & S & \end{tikzcd}$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$\begin{tikzcd} \mathcal{O}_Z(Z) & & A \arrow[ll, "f^\#"'] \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_Z(Z) & & B \arrow[ll, "g^\#"'] \\ & R \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$$\Rightarrow \quad A \otimes_R B \longrightarrow \mathcal{O}_Z(Z) \quad \Leftrightarrow \quad Z \to \operatorname{Spec} A \otimes_R B$$

$$a \otimes b \longmapsto f^\#(a) \cdot g^\#(b) \qquad\qquad \operatorname{Spec} A \otimes_R B = X \times_S Y$$

exercise: verify that this makes the appropriate diagram commutative