

To construct general fiber products, we glue the ones obtained from the affine case:

Arbitrary schemes,  $S = \bigcup_{i \in I} V_i$   $V_i = \text{Spec } R_i$

$$\pi_X^{-1}(V_i) = \bigcup_{j \in J} V_{ij}$$

$$V_{ij} = \text{Spec } A_{ij} \subset X$$

$$\pi_Y^{-1}(V_i) = \bigcup_{k \in K} W_{ik}$$

$$W_{ik} = \text{Spec } B_{ik} \subset Y$$

for  $V_{ij} \times_{R_i} W_{ik} = \text{Spec } A_{ij} \otimes_{R_i} B_{ik}$

show that these can be glued to obtain  $X \times_S Y$ .  
(details in Hartshorne) □

Example i  $A_k = \text{Spec } k[x] = \text{Spec } k[y]$

$k \hookrightarrow k[x]$  as the subring of constant polynomials.

$\Rightarrow A_k^1 \longrightarrow \text{Spec } k$  the "structural morphism".

For  $n$   $A_k^1 \times_k A_k^1 \longrightarrow A_k^1 = \text{Spec } k[x]$

$\downarrow \quad \square \quad \downarrow$   
 $\text{Spec } k[y] = A_k^1 \longrightarrow \text{Spec } k$

$A_k^1 \times_k A_k^1 = \text{Spec } k[x] \otimes_k k[y] = \text{Spec } k[x, y] = A_k^2$

$k[x] \otimes_k k[y] = k[x, y].$

More generally  $A_k^n \times_k A_k^s = A_k^{n+s}$

Note:  $P_k^1 \times_k P_k^1 \not\cong P_k^2$

Note: The underlying topological space of  $X \times_S Y$  is NOT equal to the product of the underlying topological spaces of  $X$  and  $Y$ .

In fact (Ex. II.3.9 in Homework 9) even the underlying sets are not the same.

In the case of varieties over a field, then the set of closed points of a product is the product of the sets of closed points. The field also needs to be algebraically closed. (see Ex. II.3.23)

Also see Homework 1.

The topology on  $X \times_k Y$  (even for varieties / alg closed  $k$ ) is "finer" than the product topology.

Fibers of a morphism: (<sup>see</sup> Ex. II.2.7 in Homework 7)

$X$  a scheme  $x \in X$  point  $\mathcal{O}_{X,x} \supset m_x$

Def: the residue field of  $x$  is

$$k(x) := \mathcal{O}_{X,x} / m_x$$

Ex. II.2.7:  
The data of a morphism from  $\text{Spec } K$  ( $K$  a field)  
to  $X$  with image  $x$  is equivalent to the data  
of an inclusion  $k(x) \hookrightarrow K$ .

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Given  $x \in X$ , we always have  $\text{Spec } k(x) \hookrightarrow X$   
using the identity  $k(x) \hookrightarrow k(x)$ .

Def: Given a morphism of schemes  $f: X \rightarrow Y$ ,  
 let  $y \in Y$  be a point and let  $\text{Spec } k(y) \hookrightarrow Y$  be the  
 natural morphism above. The fiber of  $f$  at  $y$

is  $X_y := X \times_Y \text{Spec } k(y)$ :

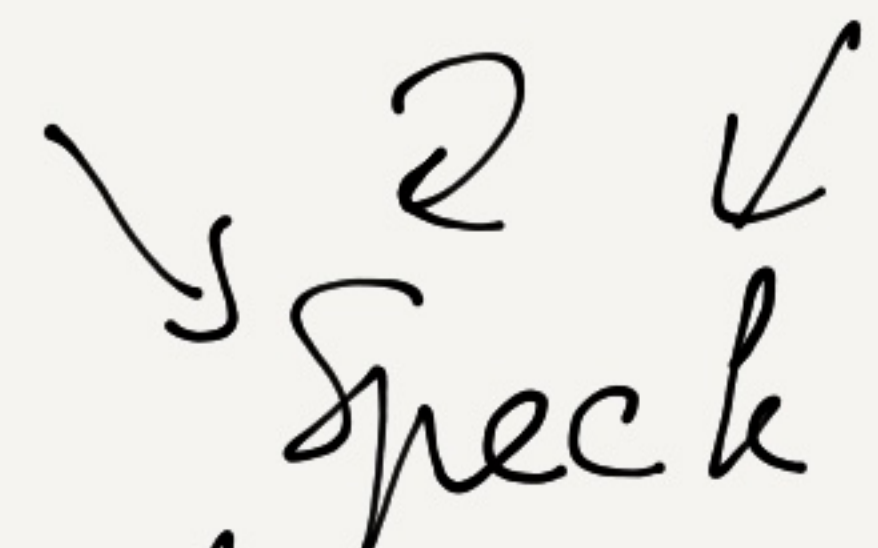
$$\begin{array}{ccc}
 X_y & \longrightarrow & X \\
 \downarrow & \square & \downarrow f \\
 \text{Spec } k(y) & \hookrightarrow & Y
 \end{array}$$

(Ex: II.3.10) The underlying topological space of  $X_y$   
 is  $f^{-1}(y) \subset X$ .

This allows us to think of  $f: X \rightarrow Y$   
as the family of schemes  $X_y$  "parametrized" by  $Y$ .

Def: Given a field  $k$ , a scheme  $X_0/k$   
(meaning  $\exists$  morphism  $X_0 \rightarrow \text{Spec } k$ )

a deformation of  $X_0$  over a scheme  $Y$  over  $k$  is  
a morphism of schemes  $X \rightarrow Y$  (over  $k$ )



s.t.  $\exists y_0 \in Y$  with  $k(y_0) = k$  and  $X_{y_0} \cong X_0$ .

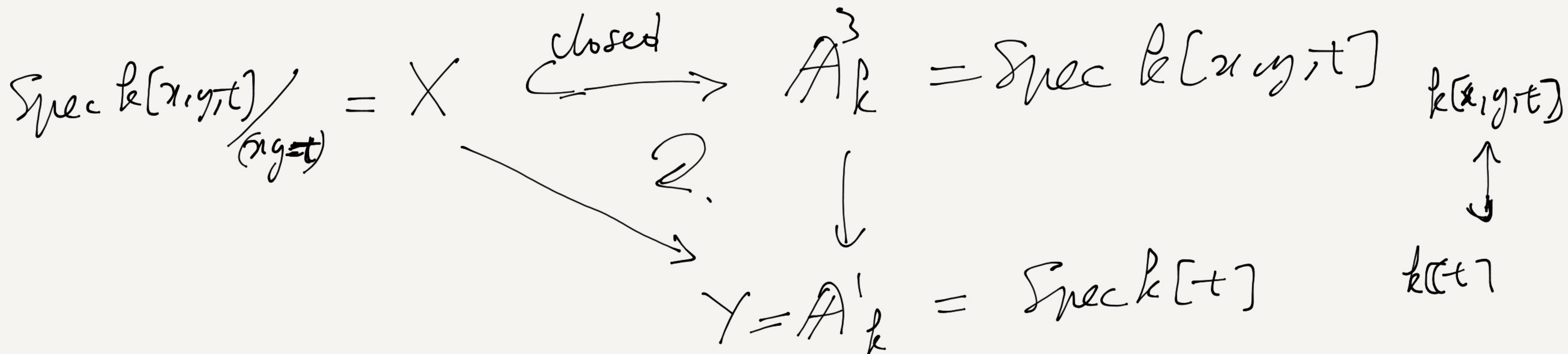
The other fibres of  $f$  are called deformations of

$X_0$ .

the "simplest" examples: (1)  $X = \text{Spec } k[x, y, t] / (xy - t)$

$$\subset \mathbb{A}_k^3$$

$$Y = \text{Spec } k[t] = \mathbb{A}_k^1$$



fibers:  $y \in Y \iff t = a \iff \text{ideal } (t - a) \in \text{Spec } k[t]$

$$\begin{aligned}
 X_y &= \text{Spec } k[x, y, t] / (xy - t) \times_Y \text{Spec } k(y) \\
 &= \text{Spec } \left( k[x, y, t] / (xy - t) \otimes_{k[t]} k(y) \right)
 \end{aligned}$$

$$y \mapsto t=a \iff (t-a) \in \text{Spec } k[t]$$

$$k(y) = \mathcal{O}_{A',y} / \mathfrak{m}_y = k[t]_{(t-a)} / (t-a)k[t]_{(t-a)}$$

$$\cong k$$

because:

$$\delta \rightarrow (t-a) \rightarrow k[t] \rightarrow k[t] / (t-a) \rightarrow 0$$

localize:

$$0 \rightarrow (t-a)_{(t-a)} \rightarrow k[t]_{(t-a)} \rightarrow (k[t] / (t-a))_{(t-a)} \rightarrow 0$$

$$\begin{array}{c} \parallel \\ (k[t] / (t-a))_{(t-a)} \\ \parallel \\ k \end{array}$$

$$\text{Spec } k(y) \hookrightarrow Y$$

$$\text{Spec } k \hookrightarrow \text{Spec } k[t]$$

$$k \longleftarrow k[t]$$

$$a \longleftrightarrow t$$



$$\begin{aligned} \Sigma_0 \quad k[x, y, t] / (xy-t) \otimes_k k(y) &= k[x, y, t] / (xy-t) \otimes_k k[t] / (t-a) \\ &\cong k[x, y, t] / (xy-t, t-a) \cong k[x, y] / (xy-a) \end{aligned}$$

So  $X_y \cong \text{Spec } k[x, y] / (xy-a)$  reducible when  $a=0$

Similarly; when  $X = \text{Spec } k[x, y, t] / (xy-x^2) \subset \mathbb{A}_k^3$   
we have

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{A}^3 = \text{Spec } k[x, y, t] \\ & \searrow & \downarrow \\ & & \mathbb{A}^1 = \text{Spec } k[t] \end{array}$$

$$\text{At } y \in Y \quad \leftrightarrow \quad t=a \quad \leftrightarrow \quad (t-a) \subset k[t]$$

we have  $X_y = \text{Spec } k[x, y] / (ay - x^2)$

non reduced when  $a=0$ .