To construct general fiber products, we glue the ones obtained from the affine case:

<u>Arbitrary schemes</u>: $S = \bigcup_{i \in I} U_i \qquad U_i = \operatorname{Spec} R_i$

$\pi_X^{-1}(U_i) = \bigcup_{j \in J} V_{ij} \qquad V_{ij} = \operatorname{Spec} A_{ij} \subset X$

$\pi_Y^{-1}(U_i) = \bigcup_{k \in K} W_{ik} \qquad W_{ik} = \operatorname{Spec} B_{ik} \subset Y$

form $V_{ij} \times_{R_i} W_{ik} = \operatorname{Spec} A_{ij} \otimes_{R_i} B_{ik}$

show that these can be glued to obtain $X \times_S Y$. $\square$

(details in Hartshorne)

<u>Example</u>: $\mathbb{A}^1_k = \operatorname{Spec} k[x] = \operatorname{Spec} k[y]$

$k \hookrightarrow k[x]$ as the subring of constant polynomials.

$\Rightarrow \quad \mathbb{A}^1_k \longrightarrow \operatorname{Spec} k$ the "structural morphism".

form

$$\begin{tikzcd} \mathbb{A}^1_k \times_k \mathbb{A}^1_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{A}^1_k = \operatorname{Spec} k[x] \arrow[d] \\ \operatorname{Spec} k[y] = \mathbb{A}^1_k \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

$$\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \operatorname{Spec} k[x] \otimes_k k[y] = \operatorname{Spec} k[x,y] = \mathbb{A}^2_k$$

$$k[x] \otimes_k k[y] = k[x,y].$$

More generally $\quad \mathbb{A}^r_k \times_k \mathbb{A}^s_k = \mathbb{A}^{r+s}_k$

<u>Note:</u> $\quad \mathbb{P}^1_k \times_k \mathbb{P}^1_k \not\cong \mathbb{P}^2_k$

Note: The underlying topological space of $X \times_S Y$ is NOT equal to the product of the underlying topological spaces of $X$ and $Y$.

In fact (Ex. II.3.9 in Homework 9) even the underlying sets are not the same.

In the case of varieties over a field, then the set of closed points of a product is the product of the sets of closed points. The field also needs to be algebraically closed. (see Ex. II.3.23)

Also see Homework 1.

The topology on $X \times_k Y$ (even for varieties / alg closed $k$) is "finer" than the product topology.

Fibers of a morphism: (see Ex. II.2.7 in Homework 7)

$X$ a scheme $\quad x \in X$ point $\quad \mathcal{O}_{X,x} \supset \mathfrak{m}_x$

Def: the residue field of $x$ is

$$k(x) := \mathcal{O}_{X,x} / \mathfrak{m}_x$$

Ex II.2.7: The data of a morphism from Spec $K$ ($K$ a field) to $X$ with image $x$ is equivalent to the data of an inclusion $k(x) \hookrightarrow K$.

Given $x \in X$, we always have $\operatorname{Spec} k(x) \hookrightarrow X$ using the identity $k(x) \hookrightarrow k(x)$.

**Def:** Given a morphism of schemes $f: X \to Y$, let $y \in Y$ be a point and let $\operatorname{Spec} k(y) \hookrightarrow Y$ be the natural morphism above. The fiber of $f$ at $y$ is

$$X_y := X \times_Y \operatorname{Spec} k(y) :$$

$$\begin{tikzcd}
X_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
\operatorname{Spec} k(y) \arrow[r, hook] & Y
\end{tikzcd}$$

(Ex: II.3.10) The underlying topological space of $X_y$ is $f^{-1}(y) \subset X$.

This allows us to think of $f: X \to Y$ as the family of schemes $X_y$ "parametrized" by $Y$.

<u>Def</u>: Given a field $k$, a scheme $X_0/k$ (meaning $\exists$ morphism $X_0 \to \operatorname{Spec} k$) a deformation of $X_0$ over a scheme $Y$ over $k$ is a morphism of schemes

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$

(over $k$)

s.t. $\exists\ y_0 \in Y$ with $k(y_0) = k$ and $X_{y_0} \cong X_0$.

The other fibers of $f$ are called deformations of $X_0$.

<u>The "simplest" examples</u>: (1) $X = \operatorname{Spec} k[x,y,t]/(xy-t)$

$$\subset \mathbb{A}^3_k$$

$$Y = \operatorname{Spec} k[t] = \mathbb{A}^1_k$$

$$\begin{tikzcd}
\operatorname{Spec} k[x,y,t]/(xy-t) = X \arrow[r, hook, "\text{closed}"] \arrow[dr] \arrow[r, phantom, shift right=4ex, "2."] & \mathbb{A}^3_k = \operatorname{Spec} k[x,y,t] \arrow[d] & k[x,y,t] \\
& Y = \mathbb{A}^1_k = \operatorname{Spec} k[t] & k[t] \arrow[u, hook]
\end{tikzcd}$$

<u>fibers</u>: $y \in Y \leftrightarrow t = a \leftrightarrow$ ideal $(t-a) \in \operatorname{Spec} k[t]$

$$X_y = \operatorname{Spec} k[x,y,t]/(xy-t) \times_Y \operatorname{Spec} k(y)$$

$$= \operatorname{Spec}\left( k[x,y,t]/(xy-t) \otimes_{k[t]} k(y) \right)$$

$$y \longleftrightarrow t = a \longleftrightarrow (t-a) \in \operatorname{Spec} k[t]$$

$$k(y) = \mathcal{O}_{\mathbb{A}^1, y} / \mathfrak{m}_y = k[t]_{(t-a)} \big/ (t-a)k[t]_{(t-a)}$$

$$\cong k$$

<u>because:</u>

$$0 \to (t-a) \to k[t] \to k[t]/(t-a) \to 0$$

$$k[t]/(t-a) = k$$

$$\Downarrow$$

localize:

$$0 \to (t-a)_{(t-a)} \to k[t]_{(t-a)} \to \left(k[t]/(t-a)\right)_{(t-a)} \to 0$$

$$\left(k[t]/(t-a)\right)_{(t-a)} = k$$

$$\left(k[t]/(t-a)\right)_{(t-a)} = k[t]_{(t-a)} \big/ (t-a)k[t]_{(t-a)} = k$$

$$\operatorname{Spec} k(y) \hookrightarrow Y$$

$$\operatorname{Spec} k(y) = \operatorname{Spec} k \hookrightarrow \operatorname{Spec} k[t]$$

$$k \twoheadleftarrow k[t]$$

$$a \longleftarrow\!\shortmid t$$

So $k[x,y,t]/(xy-t) \otimes_{k[t]} k(y) = k[x,y,t]/(xy-t) \otimes_{k[t]} k[t]/(t-a)$

$$\cong k[x,y,t]/(xy-t,\ t-a) \cong k[x,y]/(xy-a)$$

So $X_y \cong \operatorname{Spec} k[x,y]/(xy-a)$ reducible when $a=0$

Similarly, when $X = \operatorname{Spec} k[x,y,t]/(ty-x^2) \subset \mathbb{A}^3_k$.

we have

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "\varrho"'] & \mathbb{A}^3 = \operatorname{Spec} k[x,y,t] \arrow[d] \\ & \mathbb{A}^1 = \operatorname{Spec} k[t] \end{tikzcd}$$

At $y \in Y \leftrightarrow t = a \leftrightarrow (t-a) \subset k[t]$

we have $X_y = \operatorname{Spec} k[x,y] \big/ (ay - x^2)$

nonreduced when $a = 0$.