Separated and proper morphisms:

Zariski topology is not Hausdorff.

Separated $\longleftrightarrow$ Hausdorff

(replaces Hausdorff).

<u>Recall</u>: When a topology is Hausdorff, the diagonal is closed in the product topology

<u>Def</u>: Given $X \xrightarrow{\pi_X} S$, the diagonal morphism of X over S is the unique morphism

$$\Delta : X \longrightarrow X \times_S X$$

$$\begin{tikzcd} X \times_S X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi_X"] \\ X \arrow[r, "\pi_X"'] & S \end{tikzcd}$$

such that

$p_1 \circ \Delta = Id_X$

and $p_2 \circ \Delta = Id_X$

$$
\begin{tikzcd}
X \arrow[drr, bend left, "Id_X"] \arrow[dr, "\exists!\,\Delta"] \arrow[ddr, bend right, "Id_X"'] & & \\
& X \times_S X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi_X"] \\
& X \arrow[r, "\pi_X"'] & S
\end{tikzcd}
$$

Definition: A morphism $f: X \to Y$ is separated if the diagonal $\Delta: X \longrightarrow X \times_Y X$ is a closed embedding.

Lemma: Any morphism of affine schemes is separated.

Proof: $f: X \to Y \qquad X = \operatorname{Spec} A \qquad Y = \operatorname{Spec} B.$

$(\Leftarrow) \qquad f^{\#}: B \to A$

$$X \times_Y X = \operatorname{Spec} A \otimes_B A$$

$$\Delta: \begin{tikzcd} X \arrow[r] \arrow[dr, bend right, "\mathrm{Id}_X"'] \arrow[rr, bend left, "\mathrm{Id}_X"] & X \times X \arrow[r, "p_1"'] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

$$
\begin{tikzcd}
A & & \\
& A \otimes_B A \arrow[ul, "\Delta^\#"'] & A \arrow[l] \arrow[ull, bend right, "\mathrm{Id}_A"'] \\
& A \arrow[u] \arrow[uul, bend left, "\mathrm{Id}_A"] & B \arrow[l, "f^\#"] \arrow[u, "f^\#"'] \arrow[ul, phantom, "\square"]
\end{tikzcd}
$$

$a \otimes 1 \leftarrow\!\shortmid\ a$

$1 \otimes a \leftarrow\!\shortmid\ a$

$$\Delta^\#(a \otimes a') = aa'$$

$$\operatorname{Spec} A \longrightarrow \operatorname{Spec} A \otimes_B A$$

$$A \twoheadleftarrow A \otimes_B A$$

$$aa' \leftarrow\!\shortmid\ a \otimes a'$$

surjective $\Longrightarrow$

Δ is a closed embedding.

The ideal of the image of Δ is $\ker(A \otimes_B A \twoheadrightarrow A)$. $\square$

<u>Quintessential example of a non-separated morphism:</u>

k = field, $X = \mathbb{A}^1 = \operatorname{Spec} k[x] \longrightarrow \operatorname{Spec} k$

$Y = \mathbb{A}^1 = \operatorname{Spec} k[y] \longrightarrow \operatorname{Spec} k$

glue X to Y as follows:

Let $U \subset X$ be the open subscheme $\operatorname{Spec} k[x, x^{-1}]$

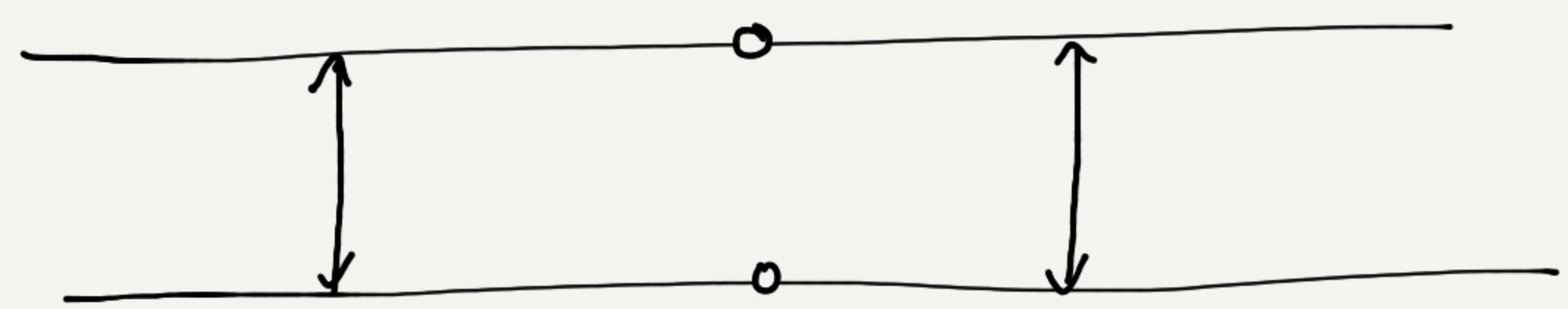
let $V \subset Y$ be the open subscheme $\operatorname{Spec} k[y, y^{-1}]$

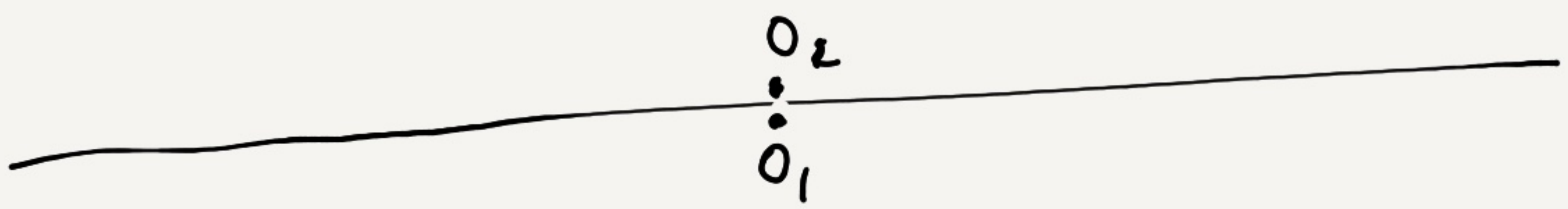
We have an isomorphism $U \xrightarrow{\varphi} V$

via $k[x, x^{-1}] \longleftarrow k[y, y^{-1}]$

$x \longleftarrow\!\shortmid\; y$

Define $Z := X \cup_{\varphi} Y$ the glueing of X and Y along φ





For simplicity, assume k is algebraically closed.

In $Z \times_{\operatorname{Spec} k} Z$, we have 4 infinitely near points:

$(0_1, 0_1)$, $(0_1, 0_2)$, $(0_2, 0_1)$, $(0_2, 0_2)$.

The image of $\Delta : Z \longrightarrow Z \times_k Z$ only contains $(0_1, 0_1)$ and $(0_2, 0_2)$, but its closure contains all four points.

$$Z \times_k Z = X \times_k X \cup X \times_k Y \cup Y \times_k X \cup Y \times_k Y$$

<u>exercise</u>: work out the details of the computation of the closure of the image of Δ.

<u>Lemma</u>: A morphism $f: X \to Y$ is separated if and only if the image of Δ is a closed subset of $X \times_Y X$.

<u>Proof</u>: If Δ is a closed embedding, its image is a closed subset of $X \times_Y X$.

Now assume the image of Δ is a closed subset. We need to show Δ is a closed embedding.

$$
\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "Id_X"] \arrow[ddr, bend right, "Id_X"'] & & \\
& X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

$\Delta(X) \subset X \times_Y X$ is closed.

$\Delta : X \longrightarrow \Delta(X)$ is a homeomorphism because

$$p_1 \circ \Delta = Id_X .$$

Next we need to see that $\Delta^{\#} : \mathcal{O}_{X \times_Y X} \longrightarrow \Delta_* \mathcal{O}_X$

is surjective. Past homework: a morphism of sheaves is surjective iff it is surjective on the stalks.

Choose $x \in X$. $\Delta^{\#}(x) : \mathcal{O}_{X \times_Y X, \Delta(x)} \longrightarrow \mathcal{O}_{X,x}$

Choose an open affine neighborhood $U = \operatorname{Spec} A$ of x in X small enough so that $f(U) \subset \underset{\text{open}}{\operatorname{Spec} B} \subset Y$

(pick $V = \operatorname{Spec} B$ first s.t. $V \ni f(x)$, then choose $U \subset f^{-1}(V)$ containing x)

$U \times_V U \subset X \times_Y X$ is an open affine neighborhood of $\Delta(x)$.

and $\Delta|_U : U \longrightarrow U \times_V U$ is a closed embedding by the previous lemma. $\mathcal{O}_{U \times_V U, \Delta(x)} = \mathcal{O}_{X \times_Y X, \Delta(x)}$

$\mathcal{O}_{U,x} = \mathcal{O}_{X,x}$

and $\left(\mathcal{O}_{U \times_V U, \Delta(x)} \to \mathcal{O}_{U,x}\right) = \left(\mathcal{O}_{X \times_Y X, \Delta(x)} \to \mathcal{O}_{X,x}\right)$

is surjective. $\square$

Valuative criteria:

Use valuation rings.

Def: K a field. A subring R of K is a valuation ring if $\forall\, x \neq 0$, $x \in K$, either $x \in R$ or $x^{-1} \in R$.

Main results about valuation rings:

1. Valuation rings are local rings.
2. Given two local subrings $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ of K, we say $(A, \mathfrak{m})$ dominates $(B, \mathfrak{n})$ if $B \subset A$

and $\quad \mathfrak{m} \cap B = \mathfrak{n}$. This defines a partial order among local subrings of K.

Valuation rings are exactly the maximal elements of the set of local subrings of K for the dominance relation.

③ Let Γ be a totally ordered abelian group. A valuation v of K with values in Γ is a map

$$v : K^* \longrightarrow \Gamma \quad \text{s.t.}$$

ⓐ $\quad v(xy) = v(x) + v(y)$

ⓑ $\quad v(x+y) \geq \min(v(x), v(y))$

The set of elements $x \in K^*$ s.t. $v(x) \geq 0 \in \Gamma$ is a valuation ring of K with maximal ideal $\{x \mid v(x) > 0\}$.