The valuative criterion for separatedness:

<u>Notation:</u> K a field, $R \subset K$ a valuation ring.

$m :=$ the maximal ideal of R

$T := \operatorname{Spec} R$ $\quad U := \operatorname{Spec} K$ one pointed scheme

$R \hookrightarrow K \quad \Rightarrow \quad \operatorname{Spec} K \hookrightarrow \operatorname{Spec} R$

$\operatorname{Spec} R$ has exactly one closed point. $\leftrightarrow m$.

has a generic point. $\leftrightarrow (0) \subset R$

$(0) \subset$ every prime ideal of $R \Rightarrow$ every point of $\operatorname{Spec} R \in \overline{\{(0)\}}$

$\Rightarrow \overline{\{(0)\}} = \operatorname{Spec} R$

$\mathcal{O}_{\operatorname{Spec} R, m} = R_m = R$, $\mathcal{O}_{\operatorname{Spec} R, (0)} = R_{(0)} = K$

<u>Theorem</u> (valuative criterion of separatedness):

Let $f : X \to Y$ be a morphism of schemes. Assume X is noetherian. Then f is separated if and only if the following condition holds. For any choice of K and R as above, and any morphisms $T \to Y$, $U \to X$ forming the commutative diagram

$$\begin{tikzcd}
(0) \arrow[d] & \operatorname{Spec} K = U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
(0) & \operatorname{Spec} R = T \arrow[r] \arrow[ur, dotted, "i"] & Y
\end{tikzcd}$$

there is at most one morphism $i : T \to X$ making the whole diagram commutative.

For the proof we first need:

<u>Lemma:</u> To give a morphism from $U = \operatorname{Spec} K$ to a scheme X is equivalent to giving a point $x_1 \in X$ and an inclusion of fields $k(x_1) \hookrightarrow K$. To give a morphism of $T = \operatorname{Spec} R$ to X is equivalent to giving two points $x_0, x_1 \in X$, with $x_0 \in Z := \overline{\{x_1\}}$, and an inclusion of fields $k(x_1) \subset K$ s.t. R dominates the local ring $\mathcal{O}_{Z,x_0}$, where Z is endowed with its reduced induced scheme structure.

<u>Proof:</u> The part about $U \to X$ was done in homework. For the second part, let $t_0 = m_R$ be the closed point of T.

and let $t_1 := (0)$ be the generic point of T.

Given a morphism $T \to X$, let x_0 and x_1 be the images of t_0 and t_1. $Z = \overline{\{x_1\}}$

Since T is reduced, $T \to X$ factors through X_{red}.

$$\begin{tikzcd} T \arrow[r] \arrow[dr] & X \\ & X_{red} \arrow[u] \end{tikzcd}$$

$$Z \subset X_{red}$$

<u>claim</u>: $T \to X$ factors through Z.

Set-theoretically $T \xrightarrow{\varphi} X$ factors through Z.

$$\begin{tikzcd} T \arrow[r, "\varphi"] \arrow[dr, "\tilde{\varphi}"'] & X \\ & Z \arrow[u, "j"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{O}_X \arrow[rr] \arrow[dr, two heads] & & \varphi_* \mathcal{O}_T \\ & j_* \mathcal{O}_Z \arrow[ur, "?"] & \end{tikzcd}$$

the factorization works when $\mathcal{O}_X \to \varphi_* \mathcal{O}_T$
is zero on the kernel of $\mathcal{O}_X \twoheadrightarrow j_* \mathcal{O}_Z$. Exercise II.3.11.

Assume this.

So we have

$$\begin{tikzcd} T \arrow[r] \arrow[dr] & X \\ & Z \arrow[u] \end{tikzcd}$$

$\Rightarrow$ local homomorphism of rings $\mathcal{O}_{Z,x_0} \to \mathcal{O}_{T,t_0}$

or $\mathcal{O}_{Z,x_0} \longrightarrow R_m = R$

this means R dominates $\mathcal{O}_{Z,x_0}$ in K.

Conversely, suppose we are given $x_0, x_1 \in X$, $x_0 \in Z = \overline{\{x_1\}}$
and the inclusion $k(x_1) \subset K$ s.t. R dominates $\mathcal{O}_{Z,x_0}$

$$\begin{tikzcd} \mathcal{O}_{Z,x_0} \arrow[r, hook] \arrow[dr, hook] & R \arrow[r, hook] & K \\ & k(x_1) \arrow[ur, hook] & \end{tikzcd}$$

<u>claim</u>: the inclusion $\mathcal{O}_{Z,x_0} \hookrightarrow R$ gives a morphism $T \to \operatorname{Spec} \mathcal{O}_{Z,x_0}$ which, composed with $\operatorname{Spec} \mathcal{O}_{Z,x_0} \to X$ gives the desired morphism $T \to X$.:

Consider an open affine neighborhood $\operatorname{Spec} A$ of x_0, then $\operatorname{Spec} A$ also contains x_1. $x_0 \in \overline{\{x_1\}}$

$$\left(x_1 \in X \setminus \operatorname{Spec} A \Rightarrow \overline{\{x_1\}} \subset X \setminus \operatorname{Spec} A \Rightarrow x_0 \in X \setminus \operatorname{Spec} A \right)$$

$Z \hookrightarrow X$ closed subscheme

$$\Rightarrow \begin{tikzcd} A_{x_0} = \mathcal{O}_{X,x_0} \arrow[r, two heads] & \mathcal{O}_{Z,x_0} \\ A \arrow[u, "\text{localization}"] \arrow[ur, bend right] & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} & T \arrow[d] & & \\ & \operatorname{Spec} \mathcal{O}_{Z,x_0} \arrow[r] \arrow[rr, bend right] & \operatorname{Spec} A \arrow[r, hook] & X \end{tikzcd}$$

$\square$

Some terminology: When $x_0 \in \overline{\{x_1\}}$ we say

- x_0 is a specialization of x_1
- x_1 is a generization of x_0.

Lemma: Let $f: X \to Y$ be a quasi-compact morphism of schemes. Then the subset $f(X)$ of Y is closed if and only if it is closed under specialization, meaning $\forall\, y \in f(X)$, any specialization of y also belongs to $f(X)$.

Proof: Read

Proof of the valuative criterion: First suppose $f: X \to Y$ is separated and suppose we have and two morphisms h, h'

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dotted, shift left, "h'"] \arrow[ur, dotted, shift right, "h"'] & Y \end{tikzcd}$$

We will show $h = h'$.

$$
\begin{tikzcd}
U \arrow[r, hook] \arrow[drrr, bend left] \arrow[ddrr, bend right] & T \arrow[drr, bend left, "h"] \arrow[dr, dashed, "{(h,h')}"] \arrow[ddr, "h'"'] & & \\
& & X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
& & X \arrow[r, "f"'] & Y
\end{tikzcd}
$$

(h,h') is the unique morphism given by the universal property of fiber products.

$h|_U = h'|_U \Rightarrow (h,h')|_U$ factors through $\Delta : X \to X \times_Y X$

$h(U) = h'(U)$ $\quad$ $(h(U), h'(U)) \in \Delta(X) \subset X \times_Y X$

$\quad$ $\shortparallel$

the generic point of the image of (h,h')

f is separated $\Rightarrow \Delta(X)$ is closed $\Rightarrow \Delta(X) \ni$ image of closed point of T.