

The valuative criterion for separatedness:

Notation: K a field, $R \subset K$ a valuation ring.

$\mathfrak{m} :=$ the maximal ideal of R

$T := \text{Spec} R$ $U := \text{Spec} K$ one pointed scheme

$R \hookrightarrow K \quad \Rightarrow \quad \text{Spec} K \hookrightarrow \text{Spec} R$

$\text{Spec} R$ has exactly one closed point. $\Leftrightarrow \mathfrak{m}$.

has a generic point. $\Leftrightarrow (0) \subset R$

$(0) \subset$ every prime ideal of $R \Rightarrow$ every point of $\text{Spec} R \in \overline{\{(0)\}}$

$\Rightarrow \overline{\{(0)\}} = \text{Spec} R$

$\mathcal{O}_{\text{Spec} R, \mathfrak{m}} = R_{\mathfrak{m}} = R$, $\mathcal{O}_{\text{Spec} R, (0)} = R_{(0)} = K$

Theorem (valuative criterion of separatedness):

Let $f: X \rightarrow Y$ be a morphism of schemes. Assume X is noetherian. Then f is separated if and only if the following condition holds. For any choice of K and R as above, and any morphisms $T \rightarrow Y$,

$U \rightarrow X$ forming the commutative diagram

$$\begin{array}{ccccc} (0) & \text{Spec } K = U & \longrightarrow & X & \\ & \downarrow & & \nearrow i & \downarrow f \\ (0) & \text{Spec } R = T & \longrightarrow & Y & \end{array}$$

there is at most one morphism $i: T \rightarrow X$ making the whole diagram commutative.

For the proof we first need:

Lemma: To give a morphism from $U = \text{Spec } k$ to a scheme X is equivalent to giving a point $x_1 \in X$ and an inclusion of fields $k(x_1) \hookrightarrow K$. To give a morphism of $T = \text{Spec } R$ to X is equivalent to giving two points $x_0, x_1 \in X$, with $x_0 \in Z := \overline{\{x_1\}}$, and an inclusion of fields $k(x_1) \subset K$ s.t. R dominates the local ring \mathcal{O}_{Z, x_0} , where Z is endowed with its reduced induced scheme structure.

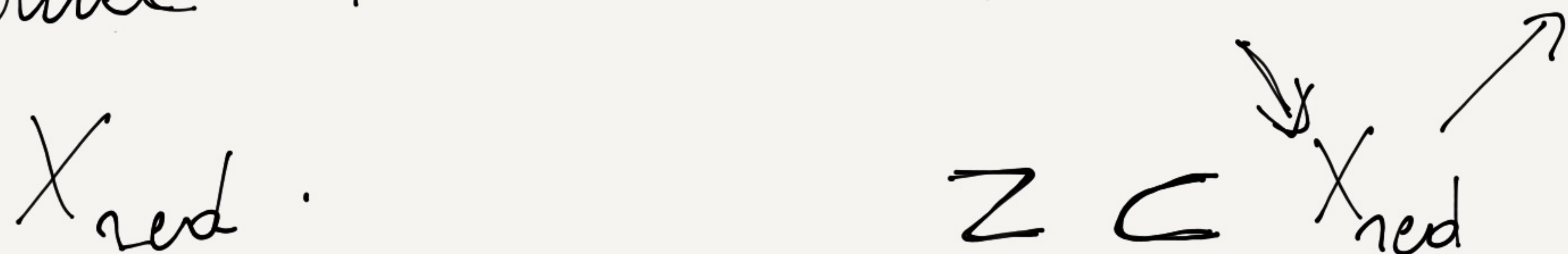
Proof: The part about $U \rightarrow X$ was done in homework.

For the second part, let $t_0 = \mathfrak{m}_R$ be the closed point of T .

and let $t_1 := (0)$ be the generic point of T .

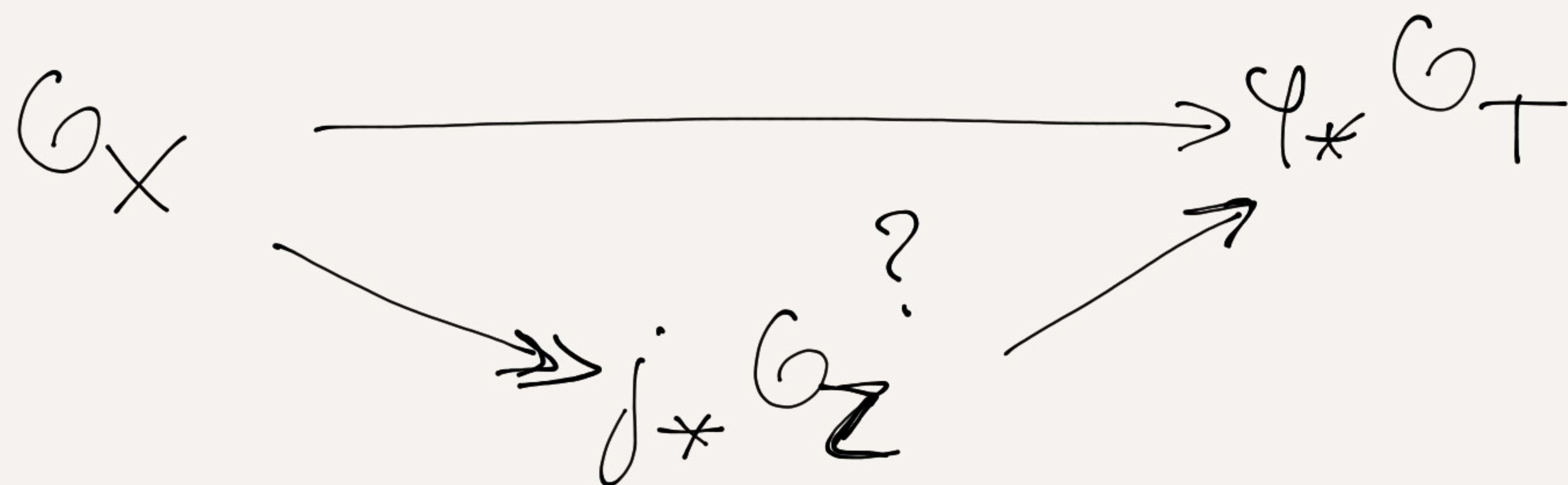
Given a morphism $T \rightarrow X$, let x_0 and x_1 be the images of t_0 and t_1 . $Z = \overline{\{x_1\}}$

Since T is reduced, $T \rightarrow X$ factors through



claim: $T \rightarrow X$ factors through Z .

Set-theoretically, $T \xrightarrow{\varphi} X$ factors through Z .



the factorization works when $\mathcal{O}_X \rightarrow \varphi_* \mathcal{O}_T$ is zero on the kernel of $\mathcal{O}_X \rightarrow j_* \mathcal{O}_Z$. Exercise II.3.11.

Assume this.

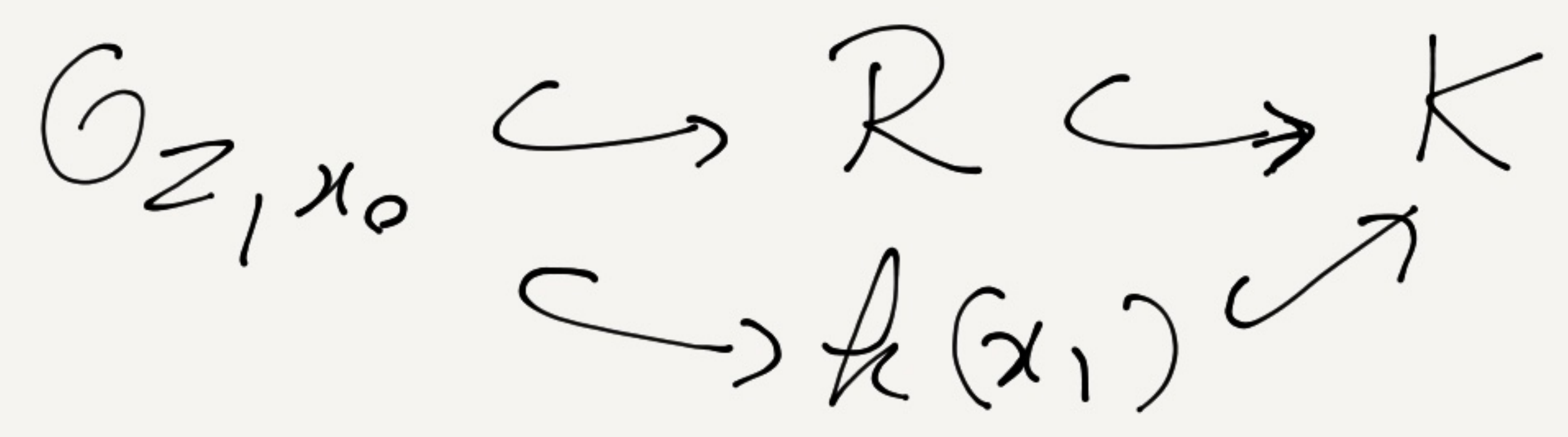


\Rightarrow local homomorphism of rings $\mathcal{O}_{Z, x_0} \rightarrow \mathcal{O}_{T, t_0}$

$\mathcal{O}_{Z, x_0} \rightarrow R_m = R$

this means R dominates \mathcal{O}_{Z, x_0} in K .

Conversely, suppose we are given $x_0, x_1 \in X$, $x_0 \in Z = \overline{\{x_1\}}$ and the inclusion $k(x_1) \subset K$ s.t. R dominates \mathcal{O}_{Z, x_0}



claim: the inclusion $\mathcal{O}_{Z, x_0} \hookrightarrow R$ gives a morphism

$T \rightarrow \text{Spec } \mathcal{O}_{Z, x_0}$ which, composed with $\text{Spec } \mathcal{O}_{Z, x_0} \rightarrow X$ gives the desired morphism $T \rightarrow X$:

Consider an open affine neighborhood $\text{Spec } A$ of x_0 ,
 then $\text{Spec } A$ also contains x_1 . $x_0 \in \overline{\{x_1\}}$

$$\left(\begin{array}{l} x_1 \in X \setminus \text{Spec } A \Rightarrow \overline{\{x_1\}} \subset X \setminus \text{Spec } A \\ \Rightarrow x_0 \in X \setminus \text{Spec } A \end{array} \right)$$

$Z \hookrightarrow X$ closed subscheme

$$\begin{array}{ccc} \Rightarrow \mathcal{O}_{X, x_0} & \twoheadrightarrow & \mathcal{O}_{Z, x_0} \\ \uparrow \text{localization} & & \uparrow \\ A_{x_0} & & A \end{array} \Rightarrow \begin{array}{ccc} T & & \\ \downarrow & & \\ \text{Spec } \mathcal{O}_{Z, x_0} & \rightarrow & \text{Spec } A \hookrightarrow X \end{array}$$

□

Some terminology: When $x_0 \in \overline{\{x_1\}}$ we say

- x_0 is a specialization of x_1 ,
- x_1 is a generalization of x_0 .

Lemma: Let $f: X \rightarrow Y$ be a quasi-compact morphism of schemes. Then the subset $f(X)$ of Y is closed if and only if it is closed under specialization, meaning $\forall y \in f(X)$, any specialization of y also belongs to $f(X)$.

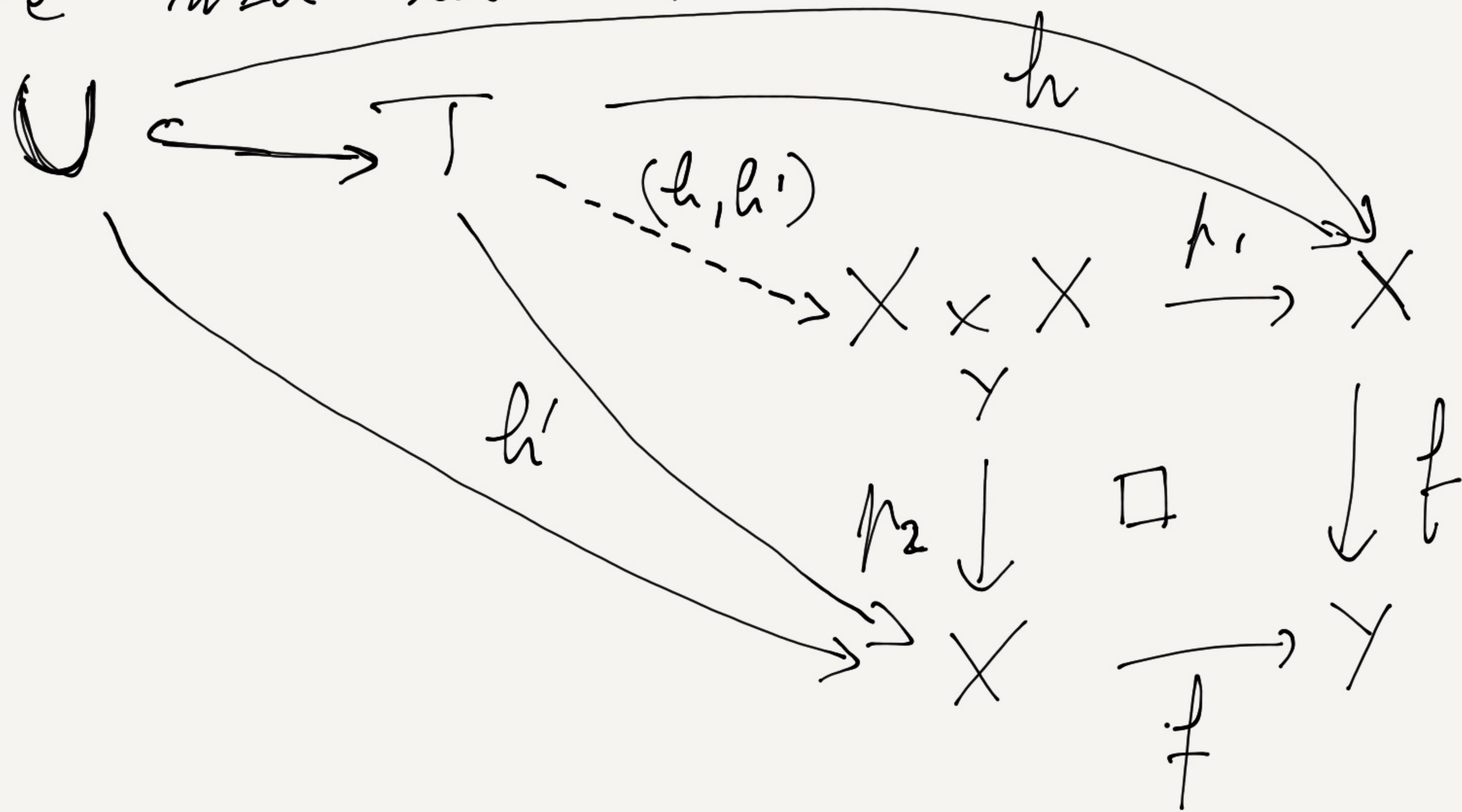
Proof: Read

Proof of the valuative criterion: First suppose $f: X \rightarrow Y$

is separated and suppose we have
and two morphisms h, h'

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \begin{array}{c} h' \cdots \rightarrow \\ \vdots \\ h \end{array} & \downarrow f \\ T & \longrightarrow & Y \end{array}$$

We will show $h = h'$.



(h, h') is the unique morphism given by the universal property of fiber products.

$$h|_U = h'|_U \implies (h, h')|_U \text{ factors through } \Delta: X \rightarrow X \times_X X$$

$$h(U) = h'(U) \implies (h(U), h'(U)) \in \Delta(X) \subset X \times_X X$$

"
the generic point of the image of (h, h')

f is separated $\implies \Delta(X)$ is closed $\implies \Delta(X) \ni$ image of closed point of T .