Note: General fact (proved using the universal property of fiber products)

a morphism $g : W \longrightarrow X \times_Y X$ factors through Δ iff $p_1 \circ g = p_2 \circ g$ (write $\Delta : X = X \times_X X \to X \times_Y X$)

So h and h' send the closed point t_0 of $T = \operatorname{Spec} R$ to the same point x_0 of X. So we have $x_0 \in \overline{\{x_1\}} = Z$

$k(x_1) = \mathcal{O}_{Z,x_1} \subset K$ given by $h|_U = h'|_U : U \to X$, $U = \operatorname{Spec} K$.

the inclusions of $k(x_1)$ in K given by h and h' are the same and the images of t_0 are the same, meaning $\mathcal{O}_{Z,x_0} \subset R$ is dominated by R.

The first lemma $\Rightarrow$ $h = h'$.

Conversely, assume that the valuative criterion is satisfied. We show that $\Delta(X)$ is closed in $X \times_Y X$.

Since X is noetherian, $\Delta : X \longrightarrow X \times_Y X$ is quasi-compact.

Using the second lemma, we show $\Delta(X)$ is closed under specialization.

Let $\xi_1 \in \Delta(X) \subset X \times_Y X$ and $\xi_0 \in Z := \overline{\{\xi_1\}}$ closure in $X \times_Y X$

We show $\xi_0 \in \Delta(X)$.

We endow Z with the reduced induced scheme structure.

$\xi_1 =$ generic point of Z (Z is an integral scheme)

$\Rightarrow \mathcal{O}_{Z,\xi_1} = \text{field} =: K$

also $\mathcal{O}_{Z,\xi_1} = k(\xi_1)$ residue field in $X \times_Y X$ (exercise)

$\mathcal{O}_{Z,\xi_0} \subset K$ local subring

$\exists$ R valuation ring of K which dominates $\mathcal{O}_{Z,\xi_0}$.
Using the first lemma, this gives a morphism

$$g : T := \operatorname{Spec} R \longrightarrow X \times_Y X$$

s.t. $g(t_1) = \xi_1$ $\quad$ $g(t_0) = \xi_0$

When we compose g with the two projections

$p_1, p_2 : X \times_Y X \longrightarrow X$, we obtain two morphisms,

say $h, h' : T \longrightarrow X$

The restrictions of h and h' to $U = \operatorname{Spec} K$ are equal because $\xi_1 \in \Delta(X)$. The valuative criterion then implies $h = h'$, i.e., $p_1 \circ g = p_2 \circ g \Rightarrow g$ factors through $\Delta \Rightarrow \xi_0 \in \Delta(X)$. ~~and~~ $\Delta(X)$ is closed $\square$

In algebraic geometry, we often change the base scheme or the field of coefficients or scalars

e.g.: given a vector space $V / \mathbb{Q}$

we obtain a vector space $/\mathbb{R}$ by tensoring with $\mathbb{R}$:

$$V \underset{\mathbb{Q}}{\otimes} \mathbb{R}$$

In geometry, tensor products correspond to fiber products.

Def: Given a scheme $\begin{array}{c} X \\ \downarrow \pi_X \\ S \end{array}$, the base change of X to a scheme $S' \xrightarrow[\varphi]{} S$ is the fiber product $X \times_S S'$:

$$\begin{tikzcd} X' := X \times_S S' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\pi_X"] \\ S' \arrow[r, "\varphi"] & S \end{tikzcd}$$

Can check: separated morphisms are stable under base change, i.e; if $X \xrightarrow{\pi_X} S$ is separated, then so is $X' \xrightarrow[p_2]{} S'$.

Proper morphisms:

Properness replaces compactness.

In topology, the image of a compact set is closed, and any closed subset of a compact space is compact.

Requiring a morphism to be closed is not strong enough (the affine line $\mathbb{A}^1_k$ would then be proper$/_k$)

So we require "universally closed":

<u>Def</u>: A morphism of schemes is closed if the image of any closed subset is closed. A morphism $f: X \to Y$ is universally closed if for any $Y' \to Y$, the base change $X \times_Y Y' \to Y'$ is closed.

<u>E.g.</u>: The affine line $\mathbb{A}^1_k \to \operatorname{Spec} k$ is NOT universally closed: base change to

$$\begin{tikzcd} \mathbb{A}^1 \times \mathbb{A}^1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

is not closed

e.g. the image of the hyperbola $xy=1$ is not closed in $\mathbb{A}^1$.

$\mathbb{P}^1_k \to \operatorname{Spec} k$ is universally closed.

Def: A morphism is proper if it is separated, of finite type and universally closed.

Theorem: The valuative criterion of properness:

Let $f : X \longrightarrow Y$ be a morphism of finite type, with X noetherian. Then f is proper if and only if, for any choice of field K and valuation ring R of K, and any morphisms $T = \operatorname{Spec} R \longrightarrow Y$, $U = \operatorname{Spec} K \longrightarrow X$ which form a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dotted, "\exists!"] & Y \end{tikzcd}$$

there is a unique morphism $T \to X$ making the whole diagram commute

All schemes noetherian

Some properties: (1) open and closed embeddings are separated, closed embeddings are proper.

(2) Compositions of $\left\{\begin{array}{l}\text{separated}\\ \text{proper}\end{array}\right.$ morphisms are $\left\{\begin{array}{l}\text{separated}\\ \text{proper}\end{array}\right.$.

(3) If $f : X \to Y$, $g : Y \to Z$

$g \circ f$ separated $\Rightarrow$ f separated.

(4) $\left\{\begin{array}{l}\text{separatedness}\\ \text{properness}\end{array}\right.$ is local on the base, i.e.,

$f : X \to Y$ is $\left\{\begin{array}{l}\text{separated}\\ \text{proper}\end{array}\right.$ $\Leftrightarrow$ $\exists$ covering $Y = \bigcup_{i \in I} V_i$

s.t. $f|_{V_i} : f^{-1}(V_i) \to V_i$

is $\left\{\begin{array}{l}\text{separated}\\ \text{proper}\end{array}\right.$ $\forall i$

(5) separated and proper morphisms are stable under base change.

(6) If $f: X \to Y$, $g: Y \to Z$ and $g \circ f$ is proper and g is separated, then f is proper.

(7) Products of separated or proper morphisms are separated or proper $\left(\begin{array}{l} f: X \to Y (\to S) \\ g: X' \to Y (\to S) \end{array} \rightsquigarrow f \times g : X \times_S X' \to Y \times_S Y' \right)$

<u>Comment on (4)</u>: If $f: X \to Y$ is separated or proper, then $\forall$ open $U \subset Y$, we can make a base change

$$\begin{tikzcd} f^{-1}(U) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ U \arrow[r, hook] & Y \end{tikzcd}$$

$\Rightarrow f: f^{-1}(U) \to U$ is separated or proper.