Proof of the valuative criterion of properness:

First assume f is proper.
Then by definition, f is separated, so the uniqueness part of the valuative criterion is satisfied by the valuative criterion for separated morphisms.
We have to show existence. $U = \operatorname{Spec} K$ $T = \operatorname{Spec} R$

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ T \arrow[r] & Y \end{tikzcd}$$

Let us do the base extension from Y to T: $\begin{tikzcd} X \times_Y T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ T \arrow[r] & Y \end{tikzcd}$

$$\Rightarrow \qquad \begin{tikzcd}
U \arrow[rr, dashed] \arrow[rrr, bend left] \arrow[ddrr, bend right] & & X \times_Y T \arrow[r, "p_1"] \arrow[dd] \arrow[ddr, phantom, "\square"] & X \arrow[dd, "f"] \\
 & \xi_1, \xi_0 \in Z \arrow[ur, hook] \arrow[dr, "p_2"] & & \\
 & t_1, t_0 & T \arrow[r] \arrow[uu, bend right] & Y
\end{tikzcd}$$

$U \to X \times_Y T$ exists and is unique by the universal property of fiber products.

Let $\xi_1 \in X \times_Y T$ be the image of U and let $Z := \overline{\{\xi_1\}}$ be the closure in $X \times_Y T$ with the reduced induced scheme structure.

f is universally closed $\Rightarrow p_2(Z) \subset T$ is closed.

U maps to the generic point of $T \Rightarrow p_2(Z) \ni$ generic point t_1 of T

$\Rightarrow p_2(Z) = T$

$\Rightarrow \quad \exists \quad \xi_0 \in Z \quad$ s.t. $p_2(\xi_0) =$ closed point t_0 of T

$\Rightarrow \quad$ local homomorphism of local rings $\mathcal{O}_{T,t_0} \to \mathcal{O}_{Z,\xi_0}$

$\mathcal{O}_{T,t_0} = R$

$\Rightarrow \quad$ local hom. $\quad R \hookrightarrow \mathcal{O}_{Z,\xi_0}$ because it corresp.:

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, hook] & \mathcal{O}_{Z,\xi_0} \arrow[d, hook] \\ K \arrow[r, equal] & K \end{tikzcd}$$

$\Rightarrow \quad \mathcal{O}_{Z,\xi_0}$ dominates R

valuation rings are maximal for the domination relation

$\Rightarrow \quad R = \mathcal{O}_{Z,\xi_0}$

Lemma 4.4 in Hartshorne which we proved earlier $\Longrightarrow$

$\exists$ morphism $T \longrightarrow X \times_Y T$ sending t_0, t_1 to ξ_0, ξ_1

Now compose with p_1 to obtain the desired morphism

$$T \longrightarrow X.$$

Conversely, suppose the valuative criterion holds.

We show f is proper.

By assumption, f is of finite type.

By the valuative criterion of separatedness, f is separated.

We need to verify that f is universally closed.

Suppose we are given $Y' \longrightarrow Y$, we have the base

change

$$\begin{tikzcd} X' := X \times_Y Y' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

we show f' is closed.

Let $Z \subset X'$ be a closed subset.

Endow Z with the reduced induced scheme structure.

$f'(Z) \subset Y'$

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

f' is of finite type

$\Rightarrow$ $f'|_Z$ is also of finite type $\Rightarrow$ it is quasi-compact.

So we can apply Lemma 4.5 which we proved earlier:

$f'(Z)$ is closed $\Leftrightarrow$ it is closed under specialization.

Let $z_1 \in Z$ be a point, $y_1 := f(z_1)$

$y_0 \in \overline{\{y_1\}}$ a specialization, we show $y_0 \in f'(Z)$.

$y_0 \in \overline{\{y_1\}} = W$ with reduced induced scheme structure.

$$\mathcal{O}_{W,y_0} \hookrightarrow \mathcal{O}_{y_1,W} = K_W = k(y_1) \text{ residue field in } Y'$$

Let $R \subset K_W$ be a valuation ring dominating $\mathcal{O}_{W,y_0}$

Apply Lemma 4.4 again to obtain morphisms forming a commutative diagram

$$\begin{tikzcd}
t_0 \in U = \operatorname{Spec} K_W \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Z \ni z_1 \arrow[d, "f'"] \\
T = \operatorname{Spec} R \arrow[r] & Y' \\
t_0, t_1 \arrow[r, mapsto] & y_0, y_1 = f'(z_1)
\end{tikzcd}$$

Compose

$$\begin{tikzcd}
U \arrow[r] \arrow[dr] & Z \arrow[r] \arrow[dr, "f'"] \arrow[d, phantom, "\circlearrowleft"] & X' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\circlearrowleft"] & X \arrow[d, "f"] \\
 & T \arrow[r] & Y' \arrow[r] & Y
\end{tikzcd}$$

The valuative criterion $\Rightarrow \exists !$ lift $T \to X$ making the diagram commutative:

$$\begin{tikzcd}
U \arrow[r] \arrow[dr] & Z \arrow[r] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
 & T \arrow[u] \arrow[ur] \arrow[urr] \arrow[r] & Y' \arrow[r] & Y
\end{tikzcd}$$

obtained from the universal property of fiber products

the generic point of T maps into Z and Z is closed $\Rightarrow$ all of T maps into Z. If z_0 is the image of t_0 via $T \to Z$, then $f'(z_0) = y_0 \in f'(Z)$. $\square$