

Proof of the valuative criterion of properness:

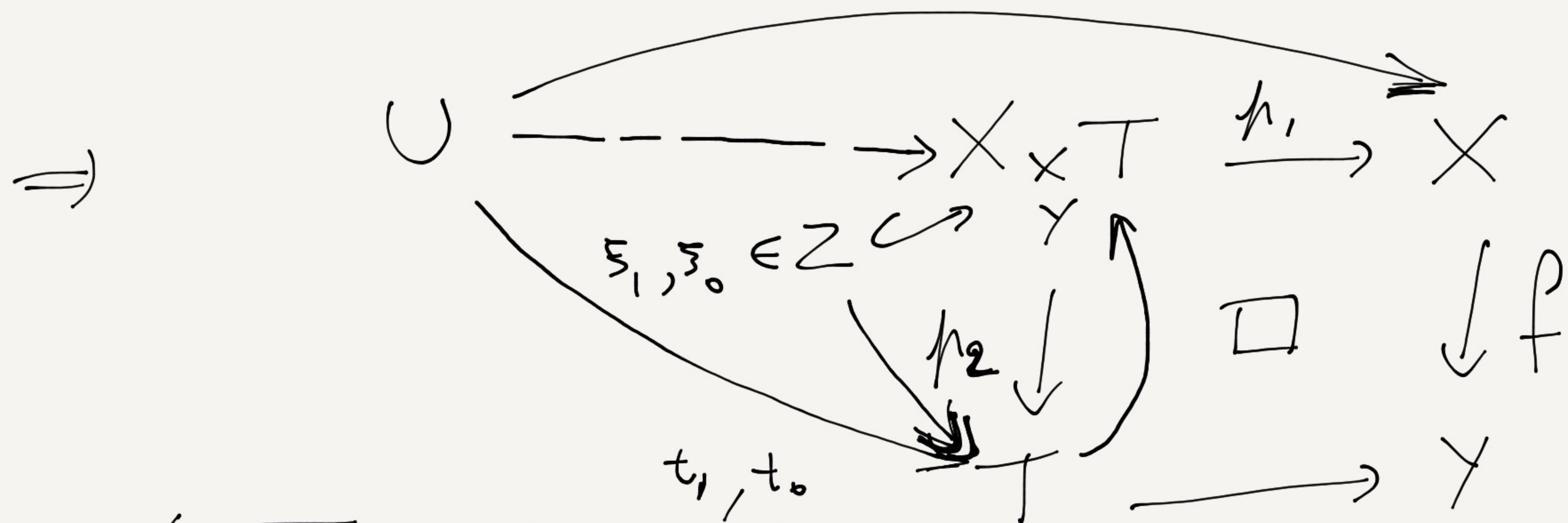
First assume f is proper.

Then by definition, f is separated, so the uniqueness part of the valuative criterion is satisfied by the valuative criterion for separated morphisms.

We have to show existence. $U = \text{Spec } K$ $T = \text{Spec } R$

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & & \downarrow f \\ T & \longrightarrow & Y \end{array}$$

Let us do the base extension from Y to T : $X \times_T Y \longrightarrow X$
 $\downarrow \quad \square \quad \downarrow f$
 $T \longrightarrow Y$



$U \rightarrow X \times_Y T$ exists and is unique by the universal property of fiber products.

Let $\xi_1 \in X \times_Y T$ be the image of U and

let $Z := \overline{\{\xi_1\}}$ be the closure in $X \times_Y T$ with the reduced induced scheme structure.

f is universally closed $\Rightarrow h_2(Z) \subset T$ is closed.

U maps to the generic point of $T \Rightarrow h_2(Z) \ni$ generic point t_1 of T

$\Rightarrow h_2(Z) = T$

$\Rightarrow \exists \xi_0 \in Z$ s.t. $\mu_2(\xi_0) =$ closed point t_0 of T

\Rightarrow local homomorphism of local rings $\mathcal{O}_{T, t_0} \rightarrow \mathcal{O}_{Z, \xi_0}$

$$\mathcal{O}_{T, t_0} = R$$

\Rightarrow local hom. $R \hookrightarrow \mathcal{O}_{Z, \xi_0}$ because it comes from:

$$\begin{array}{ccc} & \downarrow & \downarrow \\ & K & = & K \end{array}$$

$\Rightarrow \mathcal{O}_{Z, \xi_0}$ dominates R

valuation rings are maximal for the domination relation

$$\Rightarrow R = \mathcal{O}_{Z, \xi_0}$$

Lemma 4.4 in Hartshorne which we proved earlier \Rightarrow

f morphism $T \rightarrow X \times_Y^T$ sending t_0, t_1 to ξ_0, ξ_1

Now compose with ϕ_1 to obtain the desired morphism
 $T \rightarrow X$.

Conversely, suppose the valuative criterion holds.

We show f is proper.

By assumption, f is of finite type.

By the valuative criterion of separatedness, f is separated.

We need to verify that f is universally closed.

Suppose we are given $Y' \rightarrow Y$, we have the base

change $X' := X \times_Y Y' \rightarrow X$ we show f' is closed.

$$\begin{array}{ccc}
 X' := X \times_Y Y' & \longrightarrow & X \\
 \downarrow f' & \square & \downarrow f \\
 Y' & \longrightarrow & Y
 \end{array}$$

Let $Z \subset X'$ be a closed subset.

Endow Z with the reduced induced scheme structure.

$$f'(Z) \subset Y'$$

$$X' \rightarrow X$$

$$f' \downarrow \square \downarrow f$$

f' is of finite type

$$Y' \rightarrow Y$$

$\Rightarrow f'|_Z$ is also of finite type \Rightarrow it is quasi-compact.

So we can apply Lemma 4.5 which we proved earlier:

$f'(Z)$ is closed (\Rightarrow) it is closed under specialization.

Let $z_1 \in Z$ be a point, $y_1 := f(z_1)$

$y_0 \in \overline{\{y_1\}}$ a specialization, we show $y_0 \in f'(Z)$.

$y_0 \in \overline{\{y_1\}} = W$ with reduced induced scheme structure.

$$\mathcal{O}_{W, y_0} \hookrightarrow \mathcal{O}_{y_1, W} = K_W = k(y_1) \text{ residue field in } Y'$$

Let $R \subset K_W$ be a valuation ring dominating \mathcal{O}_{W, y_0} .

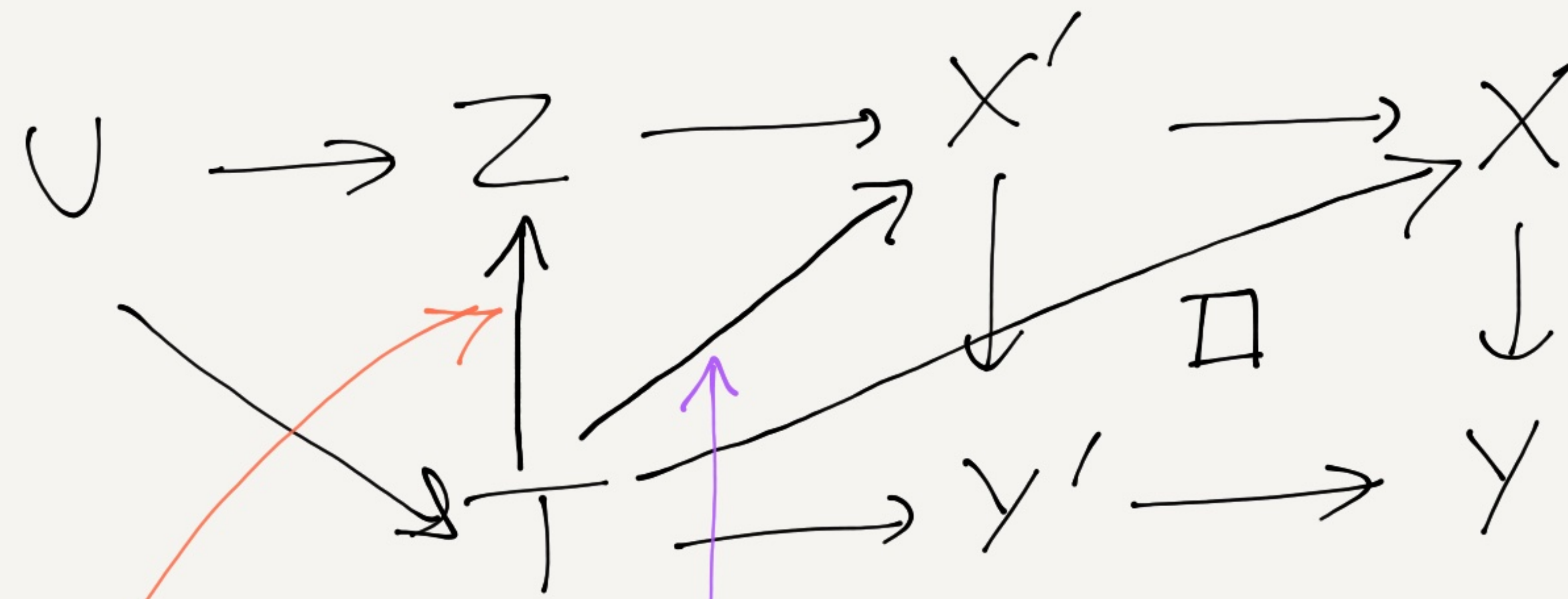
Apply Lemma 4.4 again to obtain morphisms forming a commutative diagram

$$\begin{array}{ccc} t_0 \in U = \text{Spec } K_W & \longrightarrow & Z \ni \mathcal{B}_1 \\ & \searrow \mathcal{Q} & \downarrow f' \\ T = \text{Spec } R & \longrightarrow & Y' \\ & \longmapsto & y_0, y_1 = f'(\mathcal{B}_1) \end{array}$$

Compose

$$\begin{array}{ccccccc} U & \longrightarrow & Z & \longrightarrow & X' & \longrightarrow & X \\ & \searrow & \mathcal{Q} \searrow & \downarrow f' & \mathcal{Q} & \downarrow f & \\ & & T & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$

The valuative criterion $\Rightarrow \exists!$ lift $T \rightarrow X$
 making the diagram commutative:



obtained from the universal property of fiber products

the generic point of T maps into Z and Z is closed
 \Rightarrow all of T maps into Z . If z_0 is the image of
 t_0 via $T \rightarrow Z$, then $f'(z_0) = y_0 \in f'(Z)$. \square