Sheaves of modules:

Throughout, X will denote a noetherian scheme and $\mathcal{O}_X$ will denote its structure sheaf.

(1) A sheaf of $\mathcal{O}_X$-modules is a sheaf $\mathcal{F}$ on X s.t., for all open sets $U \subset X$, the set $\mathcal{F}(U)$ is a module over the ring $\mathcal{O}_X(U)$ in such a way that the module structures are compatible with the restriction morphisms for $\mathcal{F}$ and $\mathcal{O}_X$. I.e., $\forall$ open $V \subset U \subset X$ and sections $a \in \mathcal{O}_X(U)$, $s \in \mathcal{F}(U)$, we have

$$(a s)\big|_V = a\big|_V \, s\big|_V .$$

(2) A morphism of $\mathcal{O}_X$-modules $\varphi: \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves s.t. $\varphi(U): \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is a homomorphism of $\mathcal{O}_X(U)$-modules $\forall$ $U \subset X$.

(3) The kernel, cokernel, and image of a morphism of $\mathcal{O}_X$-modules is again an $\mathcal{O}_X$-module. The quotient of two $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module (cokernel of an inclusion morphism).

(4) The direct sums, direct products, direct limits and inverse limits of $\mathcal{O}_X$-modules are again $\mathcal{O}_X$-modules. (presheaves are already sheaves, need noetherian for the direct limit, see Section 1 ex.)

(5) <u>Tensor products</u>: suppose given $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$. The tensor product $\mathcal{F} \otimes \mathcal{G}$ is the sheaf associated to the presheaf $U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$

This is an $\mathcal{O}_X$-module.

(6) <u>The sheaf $\mathcal{H}om$</u> : given two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$,

the set $\quad \mathrm{Hom}(\mathcal{F}, \mathcal{G}) := \left\{ \begin{array}{c} \text{morphisms of } \mathcal{O}_X\text{-modules} \\ \mathcal{F} \to \mathcal{G} \end{array} \right\}$.

is a module over $\mathcal{O}_X(X)$.

the sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is the sheaf associated to the presheaf

$$U \longmapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$$

the sheaf $\mathcal{H}om$ is an $\mathcal{O}_X$-module.

(7) An $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if there exists an open cover of X by affine subschemes $U = \mathrm{Spec}\, A$ s.t. $\exists$ an A-module M with $\mathcal{F}|_U \cong \tilde{M}$.

<u>Recall</u> : $\tilde{M}$ is the sheaf on $\mathrm{Spec}\, A$ s.t. $\forall$ basic open $\mathrm{Spec}\, A[f^{-1}]$, $\quad \tilde{M}(\mathrm{Spec}\, A[f^{-1}]) = M[f^{-1}]$. One proves that

then this holds for all open affine subschemes of X.

A quasi-coherent sheaf is called coherent if, in addition, M is a finite A-module.

(8) An $\mathcal{O}_X$-module is called free if it is isomorphic, as an $\mathcal{O}_X$-module, to a direct sum of sheaves isomorphic to $\mathcal{O}_X$ (sometimes also called trivial).

$\mathcal{F}$ is locally free if X has a covering by open sets U s.t. $\mathcal{F}|_U$ is free.

An isomorphism $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus I}$ is called a trivialization of $\mathcal{F}$ on U.

The rank of a locally free sheaf is the is the number of copies of $\mathcal{O}_U$ in a trivialization, it can be finite or infinite. The rank of a locally free sheaf is constant on each connected component of X.

Locally free sheaves are quasi-coherent because trivial sheaves are. They are coherent when they have finite rank.

Glueing data: Suppose $\mathcal{F}$ is locally free of finite rank n.

$$X = \bigcup_{i=1}^{m} U_i \qquad \mathcal{F}|_{U_i} \xrightarrow[\varphi_i]{\cong} \mathcal{O}_{U_i}^{\oplus n}$$

on $U_i \cap U_j$:

$$\begin{tikzcd}
\mathcal{F}|_{U_i \cap U_j} \arrow[r, "\cong", "\varphi_i"'] \arrow[d, equal] & \mathcal{O}_{U_i \cap U_j}^{\oplus n} \arrow[d, "\cong"', "\varphi_j \circ \varphi_i^{-1}"] \\
\mathcal{F}|_{U_i \cap U_j} \arrow[r, "\cong", "\varphi_j"'] & \mathcal{O}_{U_i \cap U_j}^{\oplus n}
\end{tikzcd}$$

$$\longrightarrow \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \Longleftrightarrow \varphi_j \circ \varphi_i^{-1} : \mathcal{O}_{U_i \cap U_j}^{\oplus n} \circlearrowleft$$

$$a_{kl} : \mathcal{O}_{U_i \cap U_j} \longrightarrow \mathcal{O}_{U_i \cap U_j} \qquad a_{kl} \in \mathcal{O}_X(U_i \cap U_j)$$

the matrix is invertible : it is called a transition matrix of $\mathcal{F}$ on $U_i \cap U_j$.

(9) An invertible sheaf $\mathcal{L}$ is a locally free sheaf of rank 1. If we put $\mathcal{L}^{-1} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$, then there is a natural isomorphism $\mathcal{L} \otimes \mathcal{L}^{-1} \xrightarrow{\simeq} \mathcal{O}_X$.

More generally, there is always a homomorphism of

$\mathcal{O}_X$-modules $\quad \mathcal{F} \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$

on any U: $\quad \mathcal{F}(U) \otimes \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{O}_U) \longrightarrow \mathcal{O}_X(U)$

$$s \otimes l \longmapsto l(s)$$

at the level of presheaves, factor it through the sheaf $\mathcal{F} \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. When $\mathcal{F}$ is locally free of rank 1, we can show it is an isomorphism.

The set of invertible sheaves modulo isomorphism forms a group with identity element $\mathcal{O}_X$, called the Picard group of X.

Important example: Choose an algebraically closed field k.

$$X = \mathbb{P}^n_k = \operatorname{Proj} k[X_0, \dots, X_n]$$

For any $m \in \mathbb{Z}$, we define an invertible sheaf $\mathcal{O}_{\mathbb{P}^n}(m)$:

$$\mathbb{P}^n_k = \bigcup_{i=0}^{n} U_i \qquad U_i = \operatorname{Spec} k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]$$

$$= \{\mathfrak{p} \mid \mathfrak{p}_i \not\ni X_i\}$$

$$\mathcal{O}_{U_i}(m) = \mathcal{O}_{U_i} \underbrace{X_i^{-m}}$$

formal generator.

$$\mathcal{O}_{U_i \cap U_j} \xrightarrow{\varphi_i^{-1}} \mathcal{O}_{\mathbb{P}^n}(m)\big|_{U_i \cap U_j} \xrightarrow{\varphi_j} \mathcal{O}_{U_i \cap U_j}$$

$$\mathcal{O}_{U_{ij}} X_i^{-m} \xrightarrow[\left(\frac{X_i}{X_j}\right)^m]{=} \mathcal{O}_{U_{ij}} X_j^{-m}$$

$$\Rightarrow \quad \varphi_j \circ \varphi_i^{-1} = \text{multiplication by } \left(\frac{X_i}{X_j}\right)^m$$

$\mathcal{O}_{\mathbb{P}^n}(m)$ is obtained by glueing these sheaves.

$\Rightarrow \quad \mathcal{O}_{\mathbb{P}^n}(-m) \cong \mathcal{O}_{\mathbb{P}^n}(m)^{-1} = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}(m), \mathcal{O}_{\mathbb{P}^n})$

<u>Claim</u>: In general, if $\mathcal{L}$ has trivializations

$\mathcal{L}|_{U_i} \xrightarrow{\varphi_i} \mathcal{O}_{U_i}$ with transition functions

$f_{ij} = \varphi_j \circ \varphi_i^{-1}$ on $U_i \cap U_j$, then $\mathcal{L}^{-1}$ has

trivializations $\mathcal{L}^{-1}|_{U_i} \xrightarrow{\psi_i} \mathcal{O}_{U_i}$ with transition

functions $g_{ij} = (f_{ij})^{-1} \in \mathcal{O}_{U_i \cap U_j}$.

<u>In fact</u>: $\{\mathcal{O}_{\mathbb{P}^n}(m), m \in \mathbb{Z}\} \cong \mathbb{Z}$.

To show this we will in fact show

$$\Gamma\left(\mathcal{O}_{\mathbb{P}^n}(m)\right) \longleftrightarrow \left\{\begin{array}{l}\text{homogeneous polynomials of} \\ \text{degree } m \text{ in } X_0, \dots, X_n\end{array}\right\} \cup \{0\}$$

<u>Proof</u>: $s \in \Gamma\left(\mathcal{O}_{\mathbb{P}^n}(m)\right)$

for each i, $s|_{U_i} \in \Gamma\left(\mathcal{O}_{\mathbb{P}^n}(m)|_{U_i}\right) = \Gamma\left(\mathcal{O}_{U_i} \cdot X_i^{-m}\right)$

$$s|_{U_i} \longleftrightarrow \Gamma(\mathcal{O}_{U_i}) = k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]$$

$$s|_{U_i} \longleftrightarrow f_i\left(\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right) \text{ polynomial}$$

$$\| $$

$$\frac{P_i(X_0, \dots, X_n)}{X_i^m} \longleftarrow \text{homogeneous of degree } m$$