

To see this, choose $\text{Spec } A \subset X$ s.t. $Y \cap \text{Spec } A \neq \emptyset$

Then $Y \cap \text{Spec } A$ is a closed subscheme of $\text{Spec } A$ of codim. 1 and is integral.

$y \Leftrightarrow p \subset A$ prime of height 1.

$$0 \rightarrow p \rightarrow A \rightarrow A/p \rightarrow 0$$
$$\Rightarrow 0 \rightarrow pA_p \rightarrow A_p \rightarrow (A/p)_p \stackrel{\text{residue field of } y}{\rightarrow} 0.$$

$\Rightarrow \mathcal{O}_{X,y} \stackrel{\parallel}{\rightarrow}$ has dimension 1

(all chains of prime ideals of $\mathcal{O}_{X,y}$ are contained in its maximal ideal $pA_p \Rightarrow$ there is only one chain: $0 \subset pA_p$)

We conclude that integral Weil divisors on X are in one-to-one correspondence with the points of X whose local ring has dimension 1.

Df: A Weil divisor on X is an integer linear combination of integral Weil divisors.

$$\sum_{\text{finite}} n_Y [Y]$$

$\gamma \subset X$
integral closed of codim. 1.

the set of all Weil divisors is the free abelian group with basis $\{Y \mid Y \text{ codim. 1 in } X\}$

Example: If X has dimension 1, $\gamma \subset X$ integral
of codim. 1 is a closed point. (assume X is of finite type / field k)

$$\text{a divisor} = \sum_{P \in X \text{ closed point.}} n_P [P]$$

A useful result: Nakayama's Lemma.

Def: Given a ring A , the Jacobson radical of A is the intersection of all the maximal ideals of A .

Proposition (Nakayama's lemma): A ring, $\mathcal{O} \subset A$ an ideal contained in the Jacobson radical of A , M a finitely generated A -module.

If $\mathcal{O}M = M$, then $M = 0$.

Consequence 1: If $N \subset M$ is a submodule,

$$M = \mathcal{O}M + N \Rightarrow M = N$$

Consequence 2: If A is a local ring with maximal ideal m and x_1, \dots, x_m are elements of M whose images generate

the vector space $M/\mathfrak{m}M$, then x_1, \dots, x_m generate M .

We can do a lot of useful constructions with Weil divisors if the one-dimensional local rings of X are "regular".

Def.: A finite-dimensional local ring R with maximal ideal \mathfrak{m} is regular if $\dim R$ is equal to the minimal number of generators of \mathfrak{m} .

Equivalently, by Nakayama's lemma, (if $k := R/\mathfrak{m}$)

$$\dim R = \dim_k \mathfrak{m}/\mathfrak{m}^2 \quad (\mathfrak{m}/\mathfrak{m}^2 \cong M_R^{\otimes R/\mathfrak{m}})$$

Def.: A discrete valuation ring is a noetherian regular local ring of dim. 1.

Facts: If R is a DVR, then $\mathfrak{m} = (\pi)$ principal.

Any ideal of R is a power of \mathfrak{m} . A generator of \mathfrak{m} is called a uniformizer.

For any $f \in R$, $\exists n \geq 0$ and $u \in R^\times$ s.t.

$$f = u\pi^n$$

Def: The integer n above is the valuation $v_R(f)$.

$\forall g \in K = \text{Frac}(R) \quad \exists u \in R^\times$ s.t. $g = u\pi^{v_R(g)}$

$v_K : K \rightarrow \mathbb{Z}$ is the valuation of R .

$$\left(\exists a, b \in R \text{ s.t. } g = \frac{a}{b} \quad v_R(g) = v_R(a) - v_R(b) \right)$$

Assumption (X): X is Noetherian, integral, separated,
regular in codimension 1.

Def.: We say X is regular in codimension 1 if
all the 1-dim. local rings of X are regular, i.e.,
DVRs.

Denote K the function field of X . This is the
local ring of the generic point of X and the field
of fractions of all the local rings of X (take open
affine neighbourhoods of points to see this).

We refer to the elements of K as the rational
functions on X .

Linear equivalence of Weil divisors:

Def: The Weil divisor of a rational function $f \in K$, the divisor of f is

$f \in K$, the divisor of f is

$$\text{Div}(f) := \sum_{y \in X} v_y(f)[y]$$

integral Weil divisor

This is called a principal divisor.

Principal divisors form a subgroup of the group $\text{Div}(X)$ of all Weil divisors on X (because $v_y(fg^{-1}) = v_y(f) - v_y(g)$)

Lemma: $\text{Div}(f)$ is a well-defined Weil divisor.

Proof: We need to see that $v_y(f) = 0 \forall y$ except

finitely many. Let $V = \text{Spec } A \subset X$ be open $\neq \emptyset$.

Then $K = \text{Frac } A \Rightarrow \exists g, h \in A \text{ s.t. } f = \frac{g}{h}$

Replacing A by $A[h^{-1}]$, we can assume $f \in A$

$\Rightarrow \forall \eta \in \text{Spec } A \text{ of codim 1. } f \in A_\eta$

$\Rightarrow v_\eta(f) \geq 0 \quad \forall \eta \in V \text{ of codim. 1.}$

So the set of $\eta \in X$ s.t. $v_\eta(f) < 0$ is contained
in $Y := X \setminus V$. $Y \subset X$ closed

X is Noetherian $\Rightarrow Y = \text{finite union of irreducible components.}$

$Y = Y_1 \cup \dots \cup Y_n \quad Y_i \not\subset X \text{ irreducible.}$

$\Rightarrow \text{codim } Y_i \geq 1$