

\Rightarrow Any $\eta \in X$ of codim. 1 s.t. $v_\eta(f) < 0$ is the generic point of one of the Y_i .

$\Rightarrow \{ \eta \in X \mid \eta \text{ of codim. 1, } v_\eta(f) < 0 \}$ is finite.

Replace f by f^{-1} to see that

$\{ \eta \in X \mid \eta \text{ of codim. 1, } v_\eta(f) > 0 \}$ is finite.

$\Rightarrow \text{Div}(f)$ is well-defined. \square

Definition: The divisor class group $\mathcal{C}(X)$ is the quotient of the group $\text{Div}(X)$ of Weil divisors on X by the subgroup generated by principal divisors.

Def: We say two Weil divisors $D_1, D_2 \in \text{Div}(X)$ are linearly equivalent if their difference is principal, i.e.,

$$\exists f \in K \text{ s.t. } D_1 - D_2 = \text{Div}(f).$$

Prop.: If $X = \text{Spec} A$ (in particular A is a Noetherian integral domain), then A is a UFD if and only if X is normal (i.e., A is integrally closed) and $\mathcal{C}(X) = 0$.

(Proof is Hartshorne)

Corollary: If $X = \mathbb{A}_k^n$, then $\mathcal{C}(X) = 0$

Divisors in projective space: $S := k[x_0, \dots, x_n] = \bigoplus_{d \geq 0} S_d$

$X = \text{Proj} S = \mathbb{P}_k^n$. Let $s \in S_d$, put $Y := Z(s)$.

Lemma: Y is a closed subscheme of pure codimension 1 in X (i.e., every irreducible component of Y has codim. 1).

The irreducible components of (the underlying top. space of) Y are the zeros of the irreducible factors of s .

Proof: The ideal sheaf \mathcal{I}_Y is the image of

$$s: \mathcal{O}_X(-d) \rightarrow \mathcal{O}_X.$$

Hence, in each open set $U_i = \text{Spec } S[x_i^{-1}]_0$,

the ideal sheaf $\mathcal{I}_{Y \cap U_i} = \mathcal{I}_Y|_{U_i} = \mathcal{O}_{U_i} \frac{s}{x_i^d}$

$$\text{and } \Gamma_{Y \cap U_i} = H^0(\mathcal{I}_{Y \cap U_i}) = S[x_i^{-1}]_0 \cdot \frac{s}{x_i^d}$$

$$\subset S[x_i^{-1}]_0$$

(recall $\tilde{\mathcal{I}}_{Y \cap U_i} = \mathcal{I}_{Y \cap U_i}$)

The homogeneous ideal of Y , $I_Y = \bigoplus_{e \geq 0} I_{Y,e}$

$$I_{Y,e} = \{t \in S_e \mid \mathcal{J}_Z(t) \subset \mathcal{J}_Y\}$$

on V_i , this means $\forall t \in I_{Y,e}$,

$$\frac{t}{x_i^e} \in I_{Y \cap V_i} = S[x_i^{-1}]_0 \frac{s}{x_i^d}.$$

$\Rightarrow \frac{t}{x_i^e}$ is a multiple of $\frac{s}{x_i^d}$

$\exists P_i$ hom. of degree $n_i \geq 0$ s.t.

$$\frac{t}{x_i^e} = \frac{P_i}{x_i^{n_i}} \frac{s}{x_i^d}$$

$$\Rightarrow t = P_i x_i^{d+n_i-e} s \quad \forall i$$

$\Rightarrow t$ is a multiple of s . (exercise)

$$\Rightarrow I_Y = \bigoplus_{e \geq 0} I_{Y,e} = Ss \quad \left(\begin{array}{l} = \text{ideal generated} \\ \text{by } s \end{array} \right)$$

$s \in S_d$

Write $s = s_1^{m_1} \dots s_i^{m_i}$ where s_i is irreducible of degree d_i and s_i, s_j are not proportional for $i \neq j$.

$$Z(s) = \bigcup Z(s_i) \text{ as a set. } (I_{Z(s_i)} = Ss_i)$$

$Z(s_i)$ is irreducible of codimension 1 in X . □

We determine $d(\mathbb{P}_k^n = X)$:

General Lemma: (X is not necessarily projective space)

Suppose $U \subset X$ is non empty open and let Z_1, \dots, Z_n be the codimension 1 irreducible components of $X \setminus U$.

Then, intersecting divisors of X with U produces the exact sequence:

$$0 \rightarrow \mathbb{Z}[z_1] \oplus \dots \oplus \mathbb{Z}[z_n] \rightarrow \text{Div}(X) \rightarrow \text{Div}(U) \rightarrow 0$$

which induces the exact sequence:

$$\mathbb{Z}[z_1] \oplus \dots \oplus \mathbb{Z}[z_n] \rightarrow \mathcal{O}(X) \rightarrow \mathcal{O}(U) \rightarrow 0$$

Proof: The first exact sequence is immediate: a divisor of U is the intersection with U of its closure in X . For the second sequence, surjectivity is true for the same reason. For exactness in the middle; note that

$$K = K(X) = K(U)$$

$$\text{For } f \in K, (\text{Div}_X(f)) \cap U = \text{Div}_U(f).$$

$$\begin{array}{ccccccc}
& & 0 & & 0 & & 0 \\
& & \downarrow & & \downarrow \text{Div}(f) & \longrightarrow & \text{Div}(U) & \downarrow \\
0 & \longrightarrow & \{ \text{Div}(f) \text{ supported} & \longrightarrow & \text{Prin}(X) & \longrightarrow & \text{Prin}(U) & \longrightarrow 0 \\
& & \text{on } X \setminus U \} & & & & & \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \mathbb{Z}[z_1] \oplus \dots \oplus \mathbb{Z}[z_n] & \longrightarrow & \text{Div}(X) & \longrightarrow & \text{Div}(U) & \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
& & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & \mathbb{Z}[z_1] + \dots + \mathbb{Z}[z_n] & \longrightarrow & \mathcal{O}(X) & \longrightarrow & \mathcal{O}(U) & \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow & \\
& & 0 & & 0 & & 0 & \\
& & \text{exactly} & & & & & \\
& & \text{def.} & & & & &
\end{array}$$

If $D \in \text{Div}(X)$ satisfies $D \cap U = \text{Div}_U(f)$

then $D - \text{Div}_X(f) = \text{combination of } [z_1], \dots, [z_n]$

\Rightarrow the class of D in $\mathcal{O}(X) \in$ subgroup generated by $[z_1], \dots, [z_n]$.

\Rightarrow exactness in the middle

□

We use the lemma to compute $\mathcal{O}(\mathbb{P}^n)$:

Take $U = U_0$, $Z_0 := \mathbb{P}_k^n \setminus U_0$

$$\mathcal{O}(Z_0) \longrightarrow \mathcal{O}(\mathbb{P}^n) \longrightarrow \mathcal{O}(U_0) \longrightarrow 0$$

$$U_0 \cong \mathbb{A}_k^n \Rightarrow \mathcal{O}(U_0) = 0, \text{ so:}$$

$$\mathcal{O}(Z_0) \twoheadrightarrow \mathcal{O}(\mathbb{P}^n).$$

Lemma: $\mathcal{O}(Z_0) \longrightarrow \mathcal{O}(\mathbb{P}^n)$ is injective, hence

$$\mathcal{O}(Z_0) \xrightarrow{\cong} \mathcal{O}(\mathbb{P}^n).$$

Proof: Injectivity means that there are no rational functions f on \mathbb{P}^n s.t. $\text{Div}(f) = \text{multiple of } [Z_0]$.

$$\begin{aligned} \text{Choose } f \in K(\mathbb{P}^n) &= K(U_0) = \text{Frac } k\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right] \\ &= k\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \end{aligned}$$