

So there exists polynomials p, q in $\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}$

$$\text{s.t. } f = \frac{p}{q}$$

$\Rightarrow f$ is a rational function in X_0, \dots, X_n and it has degree 0 in X_0, \dots, X_n .

$\Rightarrow f = \frac{P}{Q}$ where P, Q are coprime homogeneous polynomials of the same positive degree.

(if f is constant, $v_Y(f) = 0 \quad \forall$ prime divisor Y)

Claim: $\text{Div}(f) = \text{Div}(P) - \text{Div}(Q)$

we saw before that for a homogeneous polynomial P , the homogeneous ideal of its scheme of zeros was SP : the ideal generated by P and the irreducible

components of $Z(P)$ are the zero schemes of the irreducible factors of P .

We define a Weil divisor associated to P as:

$$\text{Div}(P) := \sum_{i=1}^m n_i [Z(P_i)]$$

where $P = \prod_{i=1}^m P_i^{n_i}$ P_i irreducible.

Proof of the claim: $\text{Div}(f) = \text{Div}(P) - \text{Div}(Q)$

Choose a point $x \in X = \mathbb{P}_k^n$ of codimension 1.

$\exists i$ s.t. $x \in U_i = \text{Spec } S[x_i^{-1}] \subset X$

$x \leftrightarrow \mathfrak{p}$ prime ideal of height 1 of $k\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$

$\Rightarrow \mathfrak{p}$ is generated by an irreducible polynomial g .

we can write $g = \frac{R}{x_i^d}$ where $R \in S_{d-1}$

The local ring $\mathcal{O}_{X, x} \cong k\left[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}\right]_{\mathfrak{p}}$

$$f = \frac{P}{Q} = \frac{\frac{P}{X_i^m}}{\frac{Q}{X_i^m}}$$

the maximal ideal is generated by the image of g .

$$v_x(f) = v_{\mathfrak{p}}(f) = v_{\mathfrak{p}}\left(\frac{P}{X_i^m}\right) - v_{\mathfrak{p}}\left(\frac{Q}{X_i^m}\right)$$

$$= \# \text{ times } g \text{ occurs as a factor of } \frac{P}{X_i^m} - \# \text{ times } g \text{ occurs as a factor of } \frac{Q}{X_i^m}$$

$$= \# \text{ times } R \text{ occurs as a factor of } P$$

$$- \# \text{ times } R \text{ occurs as a factor of } Q$$

This proves the claim. \square

The claim $\Rightarrow \nexists f \in K(\mathbb{P}_k^m)$ s.t.

$\text{Div}(f)$ is a nonzero multiple of $[Z_0]$.

$$\Rightarrow \mathbb{Z}[Z_0] \xrightarrow{\cong} \mathcal{O}(\mathbb{P}_k^m)$$

□

Cartier divisors: We remove assumption (*). We only

assume X is a Noetherian scheme

Def: The sheaf of total quotient rings:

For any open affine subset $U = \text{Spec } A$ of X , let $\mathcal{K}(U)$ be the total quotient ring of A ; i.e., $\mathcal{K}(U)$ is the localization of A at the multiplicative set of non zero divisors of A . $A \hookrightarrow \mathcal{K}(U)$.

This defines the sheaf \mathcal{K}_X on X (with restriction morphisms given by localization morphisms) because affine open sets form a basis of the topology.

Def: $\mathcal{K}_X^* \subset \mathcal{K}_X$ is the subsheaf of invertible elements

$$\mathcal{O}_X^* \subset \mathcal{O}_X \quad \text{" " " " " " " "}$$

Note: The embedding $\mathcal{O}_X \subset \mathcal{K}_X$ induces $\mathcal{O}_X^* \subset \mathcal{K}_X^*$.

Def: A Cartier divisor on X is a global section of

$$\mathcal{K}_X^* / \mathcal{O}_X^*$$

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{K}_X^* \longrightarrow \mathcal{K}_X^* / \mathcal{O}_X^* \longrightarrow 0$$

Sections of $\mathcal{K}_X^*/\mathcal{O}_X^*$ can be locally lifted
to sections of \mathcal{K}_X^* .

This means: $\forall f \in (\mathcal{K}_X^*/\mathcal{O}_X^*)(X), \exists$ covering

$$X = \bigcup_{i \in I} U_i \quad \text{s.t.} \quad \forall i \quad \exists s_i \in \mathcal{K}_X^*(U_i)$$

$$\text{with: } s_i \mapsto f|_{U_i} \in (\mathcal{K}_X^*/\mathcal{O}_X^*)(U_i) \\ \Rightarrow \mathcal{K}_X^*(U_i)$$

In other words, the datum of a Cartier divisor is
equivalent to the data of an open covering $X = \bigcup_{i \in I} U_i$
and, on each U_i , of an invertible rational function
 $s_i \in \mathcal{K}_X^*(U_i)$ s.t. $\forall i, j \quad \frac{s_i}{s_j} \in \mathcal{O}_X^*(U_i \cap U_j)$
 $\subset \mathcal{K}_X^*(U_i \cap U_j)$

Def: A Cartier divisor is called principal if it is in the image of the natural map

$$H^0(\mathcal{K}_X^*) \rightarrow H^0(\mathcal{K}_X^*/\mathcal{O}_X^*)$$

In other words, it can be represented by a single invertible rational function (in the description above

$$\underline{I = \{i\}} \quad (U_i = X \quad s_i = f.)$$

We also write a Cartier divisor as

$$\{(s_i, U_i), i \in I, s_i \in \mathcal{K}_X^*(U_i)\}$$

this is not a unique representation of a Cartier divisor.

Def: Two Cartier divisors are called linearly equivalent if their difference (for the multiplicative group structure) is principal.

Relation with invertible sheaves:

To a Cartier divisor D , we can associate an invertible subsheaf $\mathcal{O}_X(D)$ of \mathcal{K}_X as follows.

Represent D with $\{(f_i, U_i)\}$. On each U_i , define

$$\mathcal{O}_X(D)|_{U_i} := \mathcal{O}_{U_i} \cdot \frac{1}{f_i} \subset \mathcal{K}_{U_i}$$

These glue to a subsheaf $\mathcal{O}_X(D)$ of \mathcal{K}_X because

on $U_i \cap U_j$ because $f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j)$

$$\mathcal{O}_{U_i \cap U_j} \cdot \frac{1}{f_i} = \mathcal{O}_{U_i \cap U_j} \cdot \frac{1}{f_j} \subset \mathcal{K}_{U_i \cap U_j}$$

A similar argument shows that $\mathcal{O}_X(D)$ does not depend on the chosen representation $\{(f_i, U_i)\}$.
 Conversely, given an invertible subsheaf \mathcal{L} of \mathcal{K}_X , we can associate a Cartier divisor to it as follows.
 Choose an open covering $\{U_i\}$ of X s.t. $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i} \otimes \mathcal{L}_i$.

Then \mathcal{L}_i we have

$$\begin{array}{ccc} \mathcal{O}_{U_i} \xrightarrow[\varphi_i]{\cong} \mathcal{L}|_{U_i} & \hookrightarrow & \mathcal{K}_{U_i} \\ \downarrow \psi & & \downarrow \psi_i^{-1} \\ 1 & \xrightarrow{\quad} & \mathcal{L}_i \end{array}$$

On $U_i \cap U_j$, $f_i/f_j \in \mathcal{O}^*(U_i \cap U_j)$

\Leftrightarrow transition function for \mathcal{L} .

So $\{(f_i, U_i)\}$ represents a Cartier divisor.

$$\mathcal{O}_{U_i \cap U_j} \xrightarrow{f_i} \mathcal{L}_{U_i \cap U_j} \xrightarrow{f_j^{-1}} \mathcal{O}_{U_i \cap U_j}$$

$$1 \longrightarrow \mathcal{F}_i^{-1} \longrightarrow \mathcal{F}_j^{-1} \longrightarrow 1$$

f_i/f_j