

$$\varphi_i : V_i \rightarrow U_i \quad \varphi_j : V_j \rightarrow U_j$$

$$\varphi_i|_{V_i \cap V_j} \stackrel{?}{=} \varphi_j|_{V_i \cap V_j}$$

$$V_i \cap V_j = \text{Spec } A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]\left[\left(\frac{x_j}{x_i}\right)^{-1}\right] \subset V_i$$

$$= \text{Spec } A\left[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}\right]\left[\left(\frac{x_i}{x_j}\right)^{-1}\right] \subset V_j$$

$$\varphi_i^\#|_{V_i \cap V_j} : \frac{x_0}{x_i} \mapsto t_{0i}, \dots, \frac{x_n}{x_i} \mapsto t_{ni}, \left(\frac{x_j}{x_i}\right)^{-1} \mapsto (t_{ji})^{-1}$$

$$\varphi_j^\#|_{V_i \cap V_j} : \frac{x_0}{x_j} \mapsto t_{0j}, \dots, \frac{x_n}{x_j} \mapsto t_{nj}, \left(\frac{x_i}{x_j}\right)^{-1} \mapsto (t_{ij})^{-1}$$

$$\varphi_i^\# \left( \frac{x_0}{x_j} \right) = \varphi_i^\# \left( \frac{x_0}{x_i} \cdot \left( \frac{x_j}{x_i} \right)^{-1} \right) = t_{0i} \cdot t_{ji}^{-1}$$

$$\text{claim: } t_{ji}^{-1} = t_{ij} \quad \text{and} \quad t_{0i} \cdot t_{ij} = t_{0j}$$

e.g., on  $V_i \cap V_j$ :  $s_j = t_{ji} s_i$  and  $s_i = t_{ij} s_j = t_{ij} t_{ji} s_i$   
 $s_i$  is a generator of a free rank 1 module  $\Rightarrow t_{ij} t_{ji} = 1$

$\varphi_i^* \mathcal{G}_{V_i} = \mathcal{G}_{V_i}$ , more generally, for any  
morphism of schemes  $\varphi: X \rightarrow Y$   $\varphi^* \mathcal{G}_Y = \mathcal{G}_X$ .  
(e.g. :  $\varphi: \text{Spec } A \rightarrow \text{Spec } B$   $N$  a  $B$  module.)

$$\Rightarrow \varphi^* \mathcal{G}_{V_i} |_{X_i} \simeq \mathcal{G}_{V_i} |_{V_i}$$

These isomorphisms glue across  $V_i \cap V_j$  because.

the transition functions of  $\mathcal{G}_{\mathbb{P}^n}(1)$  are  $(\frac{x_i}{x_j})|_{V_i \cap V_j}$ .  
which pull back to  $t_{ij}|_{V_i \cap V_j}$  by definition

$$\Rightarrow \varphi^* \mathcal{G}_{\mathbb{P}^n}(1) \cong \mathcal{L}.$$

$$\varphi^*(x_i|_{V_i}) = s_i|_{V_i} \quad \text{and} \quad \varphi^*(x_i/x_j|_{V_j}) = s_i|_{V_j}.$$

use  $\varphi^*(x_j|_{V_j}) = s_j|_{V_j}$   
and  $\varphi^*(x_i/x_j|_{V_j}) = t_{ij}$

$$s_i|_{V_j} = t_{ij} \cdot s_j|_{V_j} \text{ and } x_i|_{V_j} = \left(\frac{x_i}{x_j}\right) x_j|_{V_j}.$$

$$\Rightarrow \varphi^*(x_i|_{V_j}) = \varphi^*\left(\frac{x_i}{x_j} \cdot x_j|_{V_j}\right) = t_{ij} \cdot s_j|_{V_j} = s_i|_{V_j}.$$

$$\Rightarrow \varphi^*(x_i) = s_i \text{ globally.}$$

(2)  $\varphi: X \rightarrow \mathbb{P}^n$ , put  $\mathcal{L} := \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$  and

$$s_i := \varphi^* x_i \text{ via } u \mapsto u \otimes 1$$

$$H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \rightarrow H^0(\bar{\varphi}^* \mathcal{O}_{\mathbb{P}^n}(1)) \rightarrow H^0(\bar{\varphi}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes H^0(\mathcal{G}_X))$$

$$\rightarrow H^0(\bar{\varphi}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes_{\bar{\varphi}^* \mathcal{O}_{\mathbb{P}^n}} \mathcal{G}_X) = H^0(\varphi^* \mathcal{O}_{\mathbb{P}^n}(1)) = H^0(\mathcal{L})$$

Then, on  $U_j$  we have  $\frac{x_i}{x_j} \in \mathcal{O}_{\mathbb{P}^n}(U_j)$

put  $t_{ij} := \varphi^*\left(\frac{x_i}{x_j}\right)$ . Then  $x_i|_{U_j} = \frac{x_i}{x_j} x_j|_{U_j}$

$$\Rightarrow s_i|_{V_j} = t_{ij} s_j|_{V_j} \text{ if } V_j := \bar{\varphi}(U_j).$$

Since  $\mathcal{O}_{V_i}(1) = \mathcal{O}_{U_i} \cdot X_i|_{U_i}$ , we also have

$$\mathcal{L}|_{V_i} = \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)|_{V_i} = \varphi^* \mathcal{O}_{U_i}(1) = \varphi^* \mathcal{O}_{U_i} \cdot \varphi^*(X_i|_{U_i})$$

$$= \mathcal{O}_{V_i} \cdot s_i|_{V_i}$$

$\Rightarrow s_i$  generates  $\mathcal{L}$  on  $V_i$ .

In fact  $V_i$  = subset of  $X$  where  $s_i$  generates  $\mathcal{L}$  because if  $X_i \in$  maximal ideal at a point, then  $s_i \in$  maximal ideal at any preimage of that point.

□

In practice, collections of global sections do not usually generate, so we replace  $X$  with the open subset where they generate.

Example: Projections from one projective space to another.

Let  $L_0, \dots, L_n \in S_1$  = degree 1 part of  $S = A[x_0, \dots, x_n]$

$P := Z(L_0) \cap \dots \cap Z(L_n)$  is, by definition, a linear subspace of  $\mathbb{P}^n$ . Recall  $I_{Z(L_0)} = SL_0 \subset S$

$$\Rightarrow I_P = SL_0 + \dots + SL_n \subset S$$

Now assume  $A = k$  is a field. after possibly removing some of the  $L_i$ , we can assume they are linearly independent.

Then (see Chapter 1 Ex 2-11)  $P$  has codim.

$n+1$ , or dimension  $n-n-1$ .

After a linear change of coordinates, we can assume

$$L_0 = X_0, \dots, L_n = X_n$$

Then  $P \cong \mathbb{P}_k^{n-n-1} \subset \mathbb{P}_k^n$ .

$$\text{``Proj } k[X_{n+1}, \dots, X_n]$$

Choose  $X = \mathbb{P}_k^n$ ,  $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(1)$ ,  $s_0 = L_0 = X_0, \dots, s_n = L_n = X_n$ .

The open subset of  $\mathbb{P}^n$  where  $s_0, \dots, s_n$  generate is

$$U := U_0 \cup U_1 \cup \dots \cup U_n = \mathbb{P}^n \setminus P$$

The projection of center  $P$ :  $\xrightarrow{\text{By the theorem we have a morphism}} \varphi: U \rightarrow \mathbb{P}^n$

We often write  $\varphi: X = \mathbb{P}^n \dashrightarrow \mathbb{P}^n$  to indicate  
that  $\varphi$  is only well-defined on some open set.

This is an example of a rational map, i.e.,  
a map  $f: X \dashrightarrow Y$  which is a well-defined morphism  
on some open subset of  $X$ .

Application to quadrics: Assume  $A = k$  is a field of  
characteristic different from 2.

Suppose  $F \in S_2$  ( $S = k[x_0, \dots, x_n]$ )

The scheme of zeros  $Z(F)$  is, by definition, a quadric.  
Since the characteristic is  $\neq 2$ , we can use the  
Gram-Schmidt algorithm to write  $F = \sum_{i=0}^n L_i^2$  as a  
sum of squares of linearly independent linear form  
polynomials. The integer  $n$  is called the rank of  $F$ .

The linear space  $P = Z(L_0, \dots, L_n)$  is called the

vertex of  $Q := Z(F)$ .

If  $P$  is the projection of center  $P$ , the quadric  $Q$  is the inverse image of a quadric  $\bar{Q} \subset \mathbb{P}^n$

$$\begin{array}{ccc} \varphi : \mathbb{P}^n & \dashrightarrow & \mathbb{P}^n \text{ coordinates } y_0, \dots, y_n \\ \cup & & \downarrow \\ Q & \dashrightarrow & \bar{Q} \end{array}$$

$$\text{i.e., } \varphi^{-1}(\bar{Q}) = Q \setminus P \quad \varphi : \mathbb{P}^n \setminus P \rightarrow \mathbb{P}^n$$

$$q^* y_i = L_i \quad \bar{Q} = Z\left(\sum_{i=0}^n y_i^2\right)$$

is a quadric of maximal rank.

The vertex of  $\bar{Q}$  is empty.

We will see that  $P$  is the singular locus of  $Q$ , i.e., the locus where the local rings of  $Q$  are not regular.

Amples and very amples:

Definition: (1) An invertible sheaf  $\mathcal{L}$  on a scheme  $X$  over  $\text{Spec } A$  is called very ample if there exists a collection  $\{s_0, \dots, s_n\}$  of global sections of  $\mathcal{L}$  for which the associated morphism is an embedding.

Recall that an embedding is a morphism which induces an isomorphism with an open subscheme of a closed subscheme.

(2) An invertible sheaf  $\mathcal{L}$  on a scheme  $X$  is called ample if, for every coherent sheaf  $\mathcal{F}$  on  $X$ , there exists  $n_0 > 0$  s.t. for all  $n \geq n_0$ , the sheaf  $\mathcal{F} \otimes \mathcal{L}^n$  is generated by its global sections.

Of course ampleness is an absolute notion, while very ampleness is a relative notion. However, we have the following.

Theorem: Suppose  $X$  is of finite type over a noetherian ring  $A$  and  $\mathcal{L}$  is an invertible sheaf on  $X$ . Then  $\mathcal{L}$  is ample if and only if there exists  $m > 0$  such that

$\mathcal{L}^m$  is very ample.

(Proof in Hartshorne)

We would like to know when morphisms to projective space are embeddings.

Theorem: Given  $X/\text{Spec } A$ ,  $\mathcal{L}$ ,  $s_0, \dots, s_n \in H^0(\mathcal{L})$

generating  $\mathcal{L}$ , the associated morphism  $\varphi: X \rightarrow \mathbb{P}_A^n$  is a closed embedding if and only if,  $\forall i$ , the open set  $V_i$  where  $s_i$  generates  $\mathcal{L}$  is affine and the morphism

$$A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \rightarrow \mathcal{O}_X(V_i)$$

is injective

(proof in Hartshorne, or better yet: do it as an exercise).