

Theorem: Suppose  $A = k$  is an algebraically closed field and  $X$  is projective over  $k$ . (i.e;  $\exists$  a closed embedding of  $X$  into  $\mathbb{P}_k^m$  for some  $m > 0$ ).

Let  $\mathcal{L}$  be an invertible sheaf on  $X$ , so, ...,  $s_0$  global sections of  $\mathcal{L}$  generating  $\mathcal{L}$ ,  $\varphi$  the morphism associated to  $\mathcal{L}$ ,  $s_0, \dots, s_n$  and let  $V$  be the  $k$ -linear span of  $s_0, \dots, s_n$  in  $H^0(X, \mathcal{L})$ .

Then  $\varphi: X \rightarrow \mathbb{P}_k^n$  is a closed embedding if and only if

- (1)  $V$  separates points, i.e; for any two distinct closed points  $x, y \in X$ , there exists  $s \in V$  such that

$s(x) = 0$  and  $s(y) \neq 0$ .

(2)  $V$  separates tangent vectors, i.e; for any closed point  $x \in X$ , the induced  $k$ -linear map

$$\{s \in V \mid s_x \in m_x L_x\} \longrightarrow m_x L_x / m_x^2 L_x$$

is surjective.

$$= \{s \in V, s \text{ vanishes at } x\} = \{s \in V, s(x) = 0\}$$

(Proof in Hartshorne)

Remark: Recall that the Zariski tangent space

to  $X$  at  $x$  is  $(m_x / m_x^2)^*$  where the dual is  
with respect to  $k(x)$ , i.e.,  $\text{Hom}_{G_x}(m_x / m_x^2, k(x))$

also note:  $m_x / m_x^2 = M_x \otimes_{G_x} k(x)$

$X$  is projective/ $k$   $\Rightarrow X$  is of finite type/ $k$

$\Rightarrow$  the residue field of any closed point is a finite extension of  $k$ .

$\Rightarrow k(x) \cong k$  because  $k$  is algebraically closed.

To the Zariski tangent space  $T_x X := (m_x/m_x^2)^*$   
 $= \text{Hom}(m_x/m_x^2, k)$

and  $m_x m_x^2 / m_x^2 m_x^2 \cong m_x / m_x^2 = T_x^* X$   
the Zariski cotangent space.

To say that  $\{s \in V, s \text{ vanishes at } x\} \rightarrow m_x m_x^2 / m_x^2 m_x^2$   
means  $T_x X \hookrightarrow \{s \in V, s(x)=0\}^*$ .

## Relative Proj:

In exercise II.5.17, you saw the relative Spec of a quasi-coherent sheaf of algebras:

Given a quasi-coherent sheaf  $\mathcal{A}$  of  $\mathcal{O}_X$ -algebras, we construct  $\text{Spec}_{\mathcal{X}} \mathcal{A} \rightarrow X$  by gluing the schemes  $\text{Spec } \mathcal{A}(U) \rightarrow \text{Spec } \mathcal{O}_X(U)$  for all open affine subsets  $U \subset X$ .

We want to construct a relative Proj:

Assume now that we have a quasi-coherent sheaf  $\mathcal{S} = \bigoplus_{d \geq 0} \mathcal{S}_d$  of graded  $\mathcal{O}_X$ -algebras s.t.

$\mathcal{I}_0 = \mathcal{O}_X$ ,  $\mathcal{I}_1$  is coherent, and  $\mathcal{I}$  is locally generated by  $\mathcal{I}_1$  as an  $\mathcal{O}_X$ -algebra (i.e; the natural morphisms of  $\mathcal{O}_X$ -modules  $\text{Sym}^d \mathcal{I}_1 \rightarrow \mathcal{I}_d$  are surjective for all  $d \geq 0$ ).

Definition: The scheme  $\text{Proj}_{\mathcal{X}} \mathcal{I} \rightarrow \mathcal{X}$  is defined by glueing the schemes  $\text{Proj } \mathcal{I}(U) \rightarrow U$  for all open affine subschemes  $U \subset \mathcal{X}$ .

On each  $\text{Proj } \mathcal{I}(U)$ , we have the sheaf  $\mathcal{O}_U(1)$ .  
 These glue to give the global twisting sheaf  
 $\mathcal{O}(1)$  on  $\text{Proj}_{\mathcal{X}} \mathcal{I}$ .

Recall: One way of constructing  $G(r)$  was to define it as  $\overbrace{f(U)[\cdot]}^{\text{on } \text{Pug } f(U)}$ .

If  $V$  is affine  $\subset U_1 \cap U_2$  (both affine)

$$\text{then } f(U_1) \rightarrow f(V) \leftarrow f(U_2)$$

$$f(U_1)[\cdot] \rightarrow f(V)[\cdot] \leftarrow f(U_2)[\cdot]$$

$$\Rightarrow \overbrace{f(U_1)[\cdot]}^V = \overbrace{f(U_2)[\cdot]}^V = \overbrace{f(V)[\cdot]}^V$$

The projective bundle of a locally free sheaf of finite rank:

$X$  a noetherian scheme.  $\mathcal{E}$  locally free sheaf of rank  $n+1$  on  $X$ . In homework, you saw

$$\text{Sym } \mathcal{E} = \bigoplus_{m \geq 0} \text{Sym}^m \mathcal{E}$$

On  $U = \text{Spec } A \subset X$  where  $\mathcal{E}$  is trivial, choose a basis  $x_0, \dots, x_n$  of  $\mathcal{E}(U)$  so that

$$\mathcal{E}|_U = \mathcal{O}_U x_0 \oplus \dots \oplus \mathcal{O}_U x_n$$

then  $\text{Sym } \mathcal{E}|_U = \mathcal{O}_U[x_0, \dots, x_n]$  the polynomial algebra on  $\mathcal{O}_U$  with generators  $x_0, \dots, x_n$ .

Then  $\text{Spec}_X \text{Sym } \mathcal{E}|_U \cong \mathbb{A}^{n+1}_U \rightarrow U$

and  $\text{Proj}_X \text{Sym } \mathcal{E}|_U \cong \mathbb{P}^n_U \rightarrow U$

$$\mathcal{O}_X(U)[X_0, \dots, X_n] \leftarrow \mathcal{O}_X(U)$$

$P(\mathcal{E}) := \text{Proj}_X \text{Sym } \mathcal{E}$  is the projective bundle associated  
to  $\mathcal{E}$ . The projections  $\text{Proj} \text{Sym } \mathcal{E}|_U \rightarrow U$

glue to give the natural morphism  $P(\mathcal{E}) \xrightarrow{\pi} X$ ,  
and we also have the sheaf  $\mathcal{O}_{P(\mathcal{E})}(1)$ .

Locally, we saw:  $\Gamma(\mathbb{P}^n_U, \mathcal{O}(d)) \xleftarrow{\cong} \text{Sym}^d \mathcal{E}(U)$ .

These local identifications glue to give a natural  
isomorphism  $\bigoplus_{d \in \mathbb{Z}} \pi^* \mathcal{O}_{P(\mathcal{E})}(d) \xleftarrow{\cong} \text{Sym } \mathcal{E}$ .

Projective morphisms: A morphism  $X \rightarrow Y$  (Hartshorne)

is projective if it can be factored into a composition  $X \xrightarrow{\varphi} P_Y^n \rightarrow Y$  for some  $n$ , in such away that  $\varphi$  is a closed embedding.

(Grothendieck) A morphism  $X \rightarrow Y$  is projective if it can be factored into a composition

$$X \xrightarrow{\varphi} P \rightarrow Y$$

where  $P$  is a projective bundle over  $Y$  and  $\varphi$  is a closed embedding.

Def: A projective  $n$ -space bundle over  $Y$  is a scheme  $\pi: P \rightarrow Y$  s.t.  $\exists$  covering  $Y = \bigcup_{i \in I} U_i$  by open

affine sets  $V_i = \text{Spec } A_i$  s.t.  $\forall i$

$$\varphi_i: \pi^{-1}(V_i) \xrightarrow{\cong} \mathbb{P}_{V_i}^n$$

and for all  $i, j$ , and all open affine

$$Y = \text{Spec } A \subset V_i \cap V_j$$

the isomorphism  $A[x_0, \dots, x_n] \xleftarrow{\sim} (\varphi_j \circ \varphi_i^{-1})^\#$   
is  $A$ -linear.

Ex. II.5.18: Every vector bundle is the Spec of the  
symmetric algebra of some locally free sheaf.

However, not every projective bundle is the Proj of  
the symmetric algebra of some locally free sheaf.  
This is true for "nice" schemes, e.g., regular schemes.

The Hartshorne and Grothendieck definitions  
of projective morphisms are the same for base  
schemes  $X$  that are quasi-projective over an affine  
scheme.