Example of the cuspidal cubic:

$$C = k[x,y]/y^2 - x^3$$

$(0,0) \in \operatorname{Spec} C$ singular point.

$$\Omega^1_{C/k} = \frac{C\,dx \oplus C\,dy}{\langle d(y^2 - x^3) \rangle}$$

$d(y^2 - x^3) = 2y\,dy - 3x^2\,dx$

$$= C\,dx \oplus C\,dy \Big/ \langle 2y\,dy - 3x^2\,dx \rangle$$

Open set $D(x)$: $C = \left(k[x,y]/(y^2 - x^3) \right)[x^{-1}]$ $\quad k[x^{-1}] = k$

$$\Omega^1_{C[x^{-1}]/k[x^{-1}]} = \Omega^1_{C/k}[x^{-1}]$$

$$= \frac{C[x^{-1}]\,dx \oplus C[x^{-1}]\,dy}{(2y\,dy - 3x^2\,dx)} \cong C[x^{-1}]\,dy$$

free.

$$dx = \frac{2y}{3x^2}\,dy$$

Similarly, $\Omega^1_{C(y^2)/k}$ is free of rank 1 / C.

This means (we will see this in more detail) that C is smooth on the two open sets $D(x)$ and $D(y)$.

$$(0,0) \notin D(x) \cup D(y). \quad (\text{char.} \neq 2, 3)$$

<u>Sheafification:</u> Suppose given a scheme X over a scheme S. Cover S with open affine schemes $\operatorname{Spec} A$ and X with open affine schemes $\operatorname{Spec} B$ s.t. $\operatorname{Spec} B$ maps to $\operatorname{Spec} A$.

For each $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$, we define

$$\Omega^1_{\operatorname{Spec} B / \operatorname{Spec} A} := \widetilde{\Omega^1_{B/A}} \quad \text{quasi-coherent on } \operatorname{Spec} B.$$

These glue to a global sheaf $\Omega^1_{X/S}$ because taking relative differentials commutes with localization:

<u>Last quarter</u>: any intersection $\operatorname{Spec} A \cap \operatorname{Spec} A'$ (resp. $\operatorname{Spec} B \cap \operatorname{Spec} B'$) can be covered with affine open sets that are basic for $\operatorname{Spec} A$ and $\operatorname{Spec} A'$ (resp. $\operatorname{Spec} B$ and $\operatorname{Spec} B'$), i.e., of the form

$$\operatorname{Spec} A[f^{-1}] = \operatorname{Spec} A'[f'^{-1}]$$

$$\left(\Rightarrow \operatorname{Spec} B[f^{-1}] = \operatorname{Spec} B'[f'^{-1}]\right).$$

Then $\Omega^1_{B/A}[f^{-1}] = \Omega^1_{B[f^{-1}]/A[f^{-1}]} = \Omega^1_{B'[f'^{-1}]/A'[f'^{-1}]}$

$$= \Omega^1_{B'/A'}[f^{-1}].$$

Alternatively, we can define $\Omega^1_{X/S}$ using the diagonal embedding.

$$S = \bigcup_{j \in J} \operatorname{Spec} A_j \qquad \pi : X \longrightarrow S$$

$$\pi^{-1}(\operatorname{Spec} A_j) = \bigcup_{k \in K_j} \operatorname{Spec} B_{k_j}$$

$$X \times_S X = \bigcup \operatorname{Spec} B_{k_j} \otimes_{A_j} B_{k'_j} \overset{\text{open}}{\supset} \bigcup \operatorname{Spec} B_{k_j} \otimes_{A_j} B_{k_j}$$

$$\Delta : X \underset{\text{closed embedding}}{\hookrightarrow} \bigcup \operatorname{Spec} B_{k_j} \otimes_{A_j} B_{k_j} \underset{\text{open embedding}}{\hookrightarrow} X \times_S X$$

because $p_1 \circ \Delta = p_2 \circ \Delta = \mathrm{Id}_X$

Let $\mathcal{J}$ be the ideal sheaf of the image of the diagonal in $\bigcup \operatorname{Spec} B_{k_j} \otimes_{A_j} B_{k_j}$. We can define

$$\Omega^1_{X/S} := \Delta^*(\mathcal{J}/\mathcal{J}^2).$$

In both case, we have a differential:

$$d : \mathcal{O}_X \longrightarrow \Omega^1_{X/S} .$$

obtained from the differentials $B \longrightarrow \Omega^1_{B/A}$

<u>Properties:</u> (1) Base change:

$$\begin{tikzcd}
\operatorname{Spec} B' = \operatorname{Spec} B \otimes_A A' = X \times_S S' \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & X = \operatorname{Spec} B \arrow[d] \\
\operatorname{Spec} A' = S' \arrow[r] & S = \operatorname{Spec} A
\end{tikzcd}$$

$$\Omega^1_{B'/A'} = \Omega^1_{B/A} \otimes_B B'$$

$$\Rightarrow \quad \widetilde{\Omega^1_{B'/A'}} = \widetilde{\Omega^1_{B/A} \otimes_B B'} = p_1^* \Omega^1_{\operatorname{Spec} B / \operatorname{Spec} A}$$

$$\widetilde{\Omega^1_{B'/A'}} = \Omega^1_{\operatorname{Spec} B' / \operatorname{Spec} A'}$$

In the general case, these will glue to

$$\Omega^1_{X'/S'} \cong p_1^* \Omega^1_{X/S}$$

(2) <u>Pull-back</u>: $A \to B \to C$

$$\operatorname{Spec} C \xrightarrow[\varphi]{} \operatorname{Spec} B \longrightarrow \operatorname{Spec} A$$

$$\Omega^1_{B/A} \otimes_B C \longrightarrow \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \longrightarrow 0$$

$$\Rightarrow \quad \widetilde{\Omega^1_{B/A} \otimes C} \longrightarrow \widetilde{\Omega^1_{C/A}} \longrightarrow \widetilde{\Omega^1_{C/B}} \longrightarrow 0$$

$$\varphi^* \Omega^1_{\operatorname{Spec} B/\operatorname{Spec} A} \to \Omega^1_{\operatorname{Spec} C/\operatorname{Spec} A} \to \Omega^1_{\operatorname{Spec} C/\operatorname{Spec} B} \to 0$$

In the general case, these glue to $\left(Y \xrightarrow[\varphi]{} X \to S\right)$

$$\varphi^* \Omega^1_{X/S} \to \Omega^1_{Y/S} \to \Omega^1_{Y/X} \to 0$$

(3) Restriction to closed subschemes:

$$A \to B \to C = B/I$$

$$Y = \operatorname{Spec} C \underset{i}{\hookrightarrow} \underset{X}{\operatorname{Spec} B} \to \operatorname{Spec} A = S$$

$$I/I^2 \to \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to 0$$

Sheafify:

$$\widetilde{I/I^2} \to \widetilde{\Omega^1_{B/A} \otimes_B C} \to \widetilde{\Omega^1_{C/A}} \to 0$$

$$\mathcal{J}_Y/\mathcal{J}_Y^2 \to i^* \Omega^1_{\operatorname{Spec} B/\operatorname{Spec} A} \to \Omega^1_{\operatorname{Spec} C/\operatorname{Spec} A} \to 0$$

In general, $\mathcal{J}_Y/\mathcal{J}_Y^2 \to i^* \Omega^1_{X/S} \to \Omega^1_{Y/S} \to 0$

for $Y \underset{\text{closed}}{\hookrightarrow} X \to S$

Varieties: Definition: A variety over a field k is a reduced separated scheme of finite type over k.

Note: Hartshorne also assumes varieties to be irreducible.

Definition: A variety $/k$ is called nonsingular if all its local rings are regular local rings.

Note: Any localization of a regular local ring is again regular (Theorem II.8.14A). In other words, being regular is stable under generization.

The set of points of a scheme whose local rings are not regular is its singular locus.

The complement is the nonsingular (or regular) locus.

So the singular locus is closed under specialization and the nonsingular locus is closed under generization.

We shall see that if X is a variety over a field, then its singular locus is a proper closed subset.

<u>Theorem</u> (II.8.15) X separated of finite type over an algebraically closed field k. Then $\Omega^1_{X/k}$ is a locally free sheaf of finite rank equal to the dimension of X if and only if X is a nonsingular variety $/k$.

The same statement holds if k is not necessarily algebraically closed, provided that X is a variety.