

\Leftrightarrow the jacobian matrix has rank $n-d$ at

$$(x_1, \dots, x_n) = (0, \dots, 0)$$

$\Leftrightarrow \left(\frac{\partial f_i}{\partial x_j} (0, \dots, 0) \right)$ has rank $n-d$.
 $i \leq j \leq n, 1 \leq i \leq n$

The projective case: Now let $X \subset \mathbb{P}^n$ be a quasi-projective scheme / k alg. closed.

This means X is an open subscheme of a closed subscheme Y of \mathbb{P}^n . Let I_Y be the homogeneous ideal of Y ($I_Y = \bigoplus_{n \geq 0} H^0(\mathcal{I}_Y(n))$).

Choose homogeneous generators P_1, \dots, P_n of respective degrees d_1, \dots, d_n for I_Y .

Let $x \in X$ be a closed point, and, after a change of coordinates, assume that x is the origin of the affine chart $U_0 = D_+(X_0)$. Then $\frac{P_1}{x_0^{d_1}}, \dots, \frac{P_n}{x_0^{d_n}}$ generate the ideal of $Y \cap U_0$ in U_0 .

Since X is an open subscheme of Y , X and Y have the same local ring at x , hence they also have the same Zariski tangent or cotangent space.

We apply the result in the affine case.

The Zariski cotangent space can be identified with the quotient of the k -vector space generated by $d\left(\frac{x_1}{x_0}\right), \dots, d\left(\frac{x_n}{x_0}\right)$ by the subvector space

generated by $d\left(\frac{P_1}{X_0^{d_1}}\right), \dots, d\left(\frac{P_n}{X_0^{d_n}}\right)$

By the Leibnitz rule:

$$(*) \quad d\left(\frac{P_i}{X_0^{d_i}}\right) = \frac{X_0 dP_i - P_i dX_0}{X_0^{d_i+1}} = \frac{dP_i}{X_0^{d_i}}$$

in X because $P_i = 0$ in X .

$$dP_i = \sum_{j=0}^n \frac{\partial P_i}{\partial X_j} dX_j.$$

Consider the vector space with basis dX_0, \dots, dX_n .

$(*) \Rightarrow$ the rank of $\begin{pmatrix} \frac{\partial\left(\frac{P_i}{X_0^{d_i}}\right)}{\partial(X_j)} \\ \vdots \\ \frac{\partial\left(\frac{P_n}{X_0^{d_n}}\right)}{\partial(X_j)} \end{pmatrix}$ is equal to
the rank of $\begin{pmatrix} \frac{\partial P_i}{\partial X_j} \end{pmatrix}$ at $x \in X$.

In the vector space $\frac{kdx_0 \oplus \dots \oplus kdx_n}{(dP_1, \dots, dP_n)}$ has

dimension 1 more than

$$\frac{k d\left(\frac{x_1}{x_0}\right) \oplus \dots \oplus k d\left(\frac{x_n}{x_0}\right)}{\left(d\left(\frac{P_1}{x_0^{d_1}}\right), \dots, d\left(\frac{P_n}{x_0^{d_n}}\right)\right)}$$

Also X is nonsingular at $x \iff$

$$\left(\frac{\partial P_i}{\partial x_j} \right) \text{ has rank } n-d.$$

Zariski tangent spaces: In the affine case, we can

think of $T_x U$ as the subspace of $k^n = (kdx_1 \oplus \dots \oplus kdx_n)^*$

with equations given by $df_1(x), \dots, df_n(x)$

dx_1, \dots, dx_n are linear forms on \mathbb{k}^n :

$$(a_1, \dots, a_n) \in \mathbb{k}^n \quad dx_i(a_1, \dots, a_n) = a_i.$$

$$\begin{aligned} df_j(x)(a_1, \dots, a_n) &= \left(\sum_{i=1}^n \frac{\partial f_j(x)}{\partial x_i} dx_i \right) (a_1, \dots, a_n) \\ &= \sum_{i=1}^n \frac{\partial f_j(x)}{\partial x_i} dx_i(a_1, \dots, a_n) \\ &= \sum_{i=0}^n \frac{\partial f_j(x)}{\partial x_i} a_i \end{aligned}$$

$df_j(x)$ is also a linear form on $\mathbb{k}^n = T_x A^n$
Liebkittangent

$T_x U \subseteq T_x A^n$ is the linear subspace with
equations $df_1(x), \dots, df_r(x)$.

In the projective case, the Zariski tangent space to the affine cone over X at any point above x is the subspace of $\mathbb{k}^{n+1} = T_y \mathbb{A}^{n+1} = (\mathbb{k} dx_0 \oplus \dots \oplus \mathbb{k} dx_n)^*$ with equations $dP_1(y), \dots, dP_n(y)$.

$dP_i(y), \dots, dP_n(y)$ can be thought of linear polynomials in $n+1$ variables via:

$$dP_i(y)(a_0, \dots, a_n) = \sum_{j=0}^n \frac{\partial P_i}{\partial x_j}(y) a_j$$

$$dP_i(y) \leftrightarrow \sum \frac{\partial P_i}{\partial x_j}(y) X_j$$

Def: $\mathcal{Z}(dP_1(y), \dots, dP_n(y)) \subset \mathbb{P}^n$ is the embedded tangent space to X at $x \in X \subset \mathbb{P}^n$.

To finish the proof of Bertini's theorem, we need to

show that the set

$$B = \{ (x, H) \mid H \supset X \text{ and } x \text{ is a singular point of } X \cap H \}$$
$$\subset X \times |G_{P^d}(1)|$$

has a natural structure of closed subscheme of $X \times |G_{P^d}(1)|$

Proof: If $H = Z(f)$ $f \in H^0(G_{P^d}(1))$

and P_1, \dots, P_n are a set of generators for X , then

$I_{X \cap H}$ is generated by (f, P_1, \dots, P_n) .

We have $(x, H) \in B \Leftrightarrow$ the Zariski tangent space
to $X \cap H$ at x has
 $\dim. > d-1$ ($d = \dim X$)

By the description of the embedded tangent space to X_{n+1} , $T_x(X_{n+1})$ has $\dim > d-1$

$$\Leftrightarrow \begin{pmatrix} \frac{\partial f(x)}{\partial x_i} & \frac{\partial P_j(x)}{\partial x_i} \\ 0 & 0 \end{pmatrix}_{\substack{1 \leq j \leq r \\ 0 \leq i \leq n}} \text{ has rank } < n+1-d$$

\Leftrightarrow the minors of size $(n+1-d) \times (n+1-d)$ are zero.

These minors give polynomial equations for B .

The Euler sequence: For any ring A , we have an exact sequence :

$$0 \rightarrow \Omega_{\mathbb{P}_A^n}^1 \rightarrow \mathcal{O}_{\mathbb{P}_A^n}(-1)^{\oplus(n+1)} \rightarrow \mathcal{O}_{\mathbb{P}_A^n} \rightarrow 0$$

Proof: The sections $x_0, \dots, x_n \in H^0(\mathcal{O}_{\mathbb{P}^n_A}(1))$

generate $\mathcal{O}_{\mathbb{P}^n_A}(1)$. So the map

$$\mathcal{O}_{\mathbb{P}^n_A} \xrightarrow{\oplus(n+1)} \mathcal{O}_{\mathbb{P}^n_A}(1)$$

obtained by adding the maps $x_i : \mathcal{O}_{\mathbb{P}^n_A} \rightarrow \mathcal{O}_{\mathbb{P}^n_A}(1)$ is injective. Twisting this by $\mathcal{O}_{\mathbb{P}^n_A}(-1)$ we obtain a

injective homomorphism

$$\mathcal{O}_{\mathbb{P}^n_A}(-1) \xrightarrow{\oplus(n+1)} \mathcal{O}_{\mathbb{P}^n_A}$$

whose kernel is isomorphic to $\Omega_{\mathbb{P}^n_A}^1$ (see Theorem II. 8. 13 for details).