Proof of Pinola's proposition: Suppose we have \( \mathcal{C} \to X_U \) with \( g < q - 2 \).

Consider \( C_U \to X_U \)

Take the fiber product:

Claim: A generic abelian variety is simple.

Def: An abelian variety is called simple if it does not contain any nontrivial abelian subvarieties.

Some facts: For an integer and complex torus \( \mathbb{C}^n / Z^n \)

\[
H^{i,n}(\mathbb{C}^n / Z^n, \mathbb{Z}) \cong \bigwedge^i f^* (\mathbb{C}^n / Z^n, \mathbb{Z})
\]

Idea of proof: \( \mathbb{C}^n / Z^n \cong (S^1)^n \) use Lichtenbaum & induction.
Recall the exponential sequence:

\[ 0 \to \mathbb{Z}_A \to C_A \to C_A^\ast \to 0 \quad A \in \mathcal{A}_g \]

\[ 0 \to H^1(A, \mathbb{Z}) \to H^1(C_A) \to H^1(C_A^\ast) \xrightarrow{c_1} H^2(C_A, \mathbb{Z}) \to \cdots \]

\[ c_1([\alpha]) \in \Lambda^2 H^1(A, \mathbb{Z})^\ast \]

is an alternating form on \( H^2(A, \mathbb{Z}) \)

\[ \Lambda^2 = H^2(A, \mathbb{Z}) \]

\( \exists \) basis of \( H^1(A, \mathbb{Z}) \) (a symplectic basis) which \( c_1(\alpha) \) has matrix \( \begin{pmatrix} 0 & \text{Id} \\ -\text{Id} & 0 \end{pmatrix} \)

\( h_\alpha \) is the associated hermitian form on \( T_0 A \supset \Lambda^A_A \)

\[ \Lambda^A_A \otimes \mathbb{R} = T_0 A \quad h(x, y) := E(x, iy) + iE(x, y) \]

positive definite
If \( A > B \), then \( \Lambda_B = H_1(B, \mathbb{Z}) < H_1(A, \mathbb{Z}) = \Lambda_A \)
\[
T_0B < T_0A
\]

\[
\Lambda_C := \Lambda_B^\perp \quad \text{for } c \in H_0 \quad \Lambda_C \subset \Lambda_A
\]

\[
T_0C := T_0B^\perp \quad \text{for } c \in H_0 \quad \Lambda_C \subset T_0C \subset T_0A
\]

\[
C := T_0C \quad \subset A \quad T_0A = T_0B \oplus T_0C
\]

\[
B \times C = T_0B \oplus T_0C \quad \subset T_0A \quad \Lambda_B \oplus \Lambda_C \subset \Lambda_A \quad \text{is of finite index}
\]

\[
T_0A \quad \Lambda_A
\]

so the map \( B \times C \rightarrow A \) is an isomorphism.
Fix a finite abelian group $G$.

$$M_{i,G} := \{(\varphi, B, C) : B \in CA_i, C \in CA_{q-i}, \varphi : G \to B \times C\}$$

$$M_{i,G} \to CA_q$$

$$(\varphi, B, C) \mapsto B \times C / \text{im} \varphi$$

Any non-simple $\varphi$ belongs to the image of $M_{i,G}$ for some $G$ and some $i$. The set of non-simple $\varphi$ is a countable union of Zariski closed subsets of $CA_q$.

Let $M$ be an indecomposable component. Then

$$\dim M = \dim CA_i + \dim CA_{q-i} = \frac{i(i+1) + (q-i)(q-1-i)}{2}$$

$$= \frac{i^2 + i + q^2 - i^2 - 2qi + q - i}{2}$$
\( \dim M = \frac{2i^2 - 2q + q^2 + 9}{2} = \frac{q^2 + 9}{2} - i(q - i) \)

\[ \text{dim \, OA}_q - i(q - i) \]

\( \Rightarrow \) The generic abelian variety is simple

Back to proof of the proposition:

\( \sigma' : C' \rightarrow \text{CA}_U \quad U \subset Aq \) open

\( \Rightarrow \) So the generic a.v. of \( U \) is simple.

\[ \text{te} \in U \]

\( \sigma'_t : C'_t \rightarrow A_t \) fiber at \( t \)

\[ \begin{align*}
\uparrow &= g(C'_t) \\
g &= g(C_t)
\end{align*} \]

the image of \( C'_t \) is a curve so for \( t \) generic \( C'_t \rightarrow A_t \) the image is an abelian subvariety of \( A_t \)

\( J C'_t \rightarrow A_t \quad \text{the image is an abelian subvariety of } A_t \)

\( J C'_t \rightarrow A_t \Rightarrow \sigma'_t(C'_t) \) generates \( A_t \).
Claim: The image of \( C \) generates \( A \times V_t \).

\[ Tc_U \rightarrow cA_u \quad \text{vertical maps are proper.} \]

the image of the horizontal map is proper over \( U \) and contains all of \( cA_u \) except finitely many closed subsets. \( \Rightarrow \) the horizontal map is surjective.

\[ \sigma_0 \quad C \rightarrow cA_u \quad \text{where } C \text{ generates } A \times V_t. \]

Now fix \( B \in cA_{q-1} \), \( \nu_B := \{ t \in U: A_t \text{ is } n\text{og. to } B \times E \} \quad E \in cA_1 \)

As before, \( \nu_B \) is dense in \( U \).
$U_B =$ countable union of analytic curves in $U$.

Choose an analytic disc $\Delta$ in one of these curves.

Restrict to $\Delta$:

\[
\begin{array}{ccc}
\mathcal{C}' & \rightarrow & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{C} & \rightarrow & K \\
\end{array}
\]

Claim: the image of $\mathcal{C}'$, hence also of $\sigma'$, is a curve.

Since $q < q - 2$, by the rigidity theorem at the beginning of paper (to be moved later) the image of $\mathcal{C}_t$ in $K(B)$ does not move.

So the image of $\mathcal{C}'$ in $B$ is always the same curve, say $D$.

So $\forall t \in \Delta \quad C_t \rightarrow D$. 
Since \( C_t \rightarrow E_t \) and \( E_t \) moves with \( t \)
(because \( J : C_t \rightarrow A_t \rightarrow B \times E_t \) parameter),
the \( C_t \) are not all isomorphic.

\[ \forall t : C_t \rightarrow D \text{ so these are nontrivial ramified} \]

degree \( d \geq 2 \) covers

Riemann-Hurwitz: \[ 2g_{C_t} - 2 = d (2g_D - 2) + \deg \text{Ramification} \]

\[ d \geq 2 \implies 2g_{C_t} - 2 \geq 4(g_D - 1) + \deg \text{Ramification}. \]

\[ \implies m - 1 \geq 2g_D - 2 \implies m = 2g_D - 1 \]

\[ \implies m \geq 2g_D \geq 2 \dim B \geq 4 \]

because \( D \) generates \( B \)

because \( C_t \) generate \( A_t \).

Let \( R \subset M_Y \) be the set of all curves that admit
maps of degree \( > 2 \) to curves of genus \( \geq 2 \).