Proof: Suppose $(C, \varphi)$ is not rigid in $K$. This means $\exists$ a smooth analytic surface $S$ and a smooth proper morphism $S \rightarrow B$ to a smooth analytic curve $B$. 

$\exists \ f \in B \ s.t. \ \varphi^{-1}(f) \subseteq C$

$\exists$ a generically finite $\sigma: S \rightarrow K$ s.t. $\varphi \big|_{\sigma^{-1}(f)} = \varphi$

From the fiber product $S \times_K A \rightarrow A$

$\downarrow \quad \downarrow \sigma$

$S \rightarrow K$

$S' :=$ normalization of $S \times_K A$

$c \in S' \rightarrow A$

$\downarrow \delta \downarrow \downarrow$
We can replace $C$ by a nearby fiber: we can assume $C$ is a generic fiber of $p$. Then $C'$ is a generic fiber of $S' \to B$. Since $S'$ is normal, it is smooth in a neighborhood of $C'$.

$C' \to C$ is a double cover induced by $A \to K$ and $g < q-1$ and $C'$ generates $A$, $C'$ is irreducible.

$\Rightarrow S'$ is an irreducible surface.

\[ \Lambda^2 H^0(-\mathcal{O}^2_A) = H^0(\mathcal{O}^2_A) \to H^0(-\mathcal{O}^2_{S'}) = H^0(\omega_{S'}) \]

Restriction \[ \to H^0(\omega_{C'}) \]

Shrink $S'$ so that it becomes smooth. $C' \subset S'$

$N_{C'/S'} = C'$ because $C'$ is a fiber

adjunction: $\omega_{C'} \cong \omega_{S'} \otimes N_{C'/S'} \cong \omega_{S'} |_{C'}$
Claim: The composition $\varepsilon: \Lambda^2 H^0(\Omega^1 A) \to H^0(\omega C)$ takes image in $H^0(\omega C) \subset H^0(\omega C)$.

Proof: Since a double cover, the involution is induced by $C(-1) = A$ and $H^0(\omega C)$ is the subspace of invariants for this involution.

\[ -1 \circ \Lambda^2 H^0(\Omega^1 A) \xrightarrow{\varepsilon} H^0(\omega C) \]

\[ -1: H^0(\Omega^1 A) \to H^0(\Omega^1 A) = (\Omega^1 A)^* = -Id \]

So, $-1$ acts as $+Id$ on the image of $\varepsilon$.

\[ \dim \varepsilon \subset H^0(\omega C) \text{ has dim } g < q-1 \]
Xiao's lemma: \( \dim I_m s \geq q-1 \)
gives a contradiction.

Theorem (Xiao): \( f: S \rightarrow B \) be a proper map
map of a smooth projective surface onto a smooth proj.
curve \( B \). Suppose that we have a generically
finite map \( S \rightarrow A \) where \( A \) is an abelian variety
such the image of a generic fiber of \( f \) generates \( A \).

Then, for a general section \( w \in H^0(S^1 A) \), the
image of the map \( w \wedge H^0(S^1 A) \rightarrow H^0(w_F) \)
has \( \dim \geq q-1 \).
Proof: The map: \( \omega \wedge H^0(\Sigma^1 A) \rightarrow \wedge^2 H^0(\Sigma^1 A) = \)

\[ = H^0(L^2 A) \rightarrow H^0(\Sigma^2 S) = H^0(\mathcal{O}_S) \xrightarrow{\text{adjunction}} H^0(\mathcal{O}_F) \]

\[ N_F/S \cong \mathcal{O}_F \quad \mathcal{O}_S \otimes N_F/S \cong \mathcal{O}_F \]

Claim: the map \( \omega \wedge H^0(\Sigma^1 A) \xrightarrow{\delta} H^0(\mathcal{O}_F) \)

is injective.

i.e., \( \forall \omega \in H^0(\Sigma^1 A) \) \( \delta(\omega \wedge \omega') \neq 0 \)

Choose \( p \in F \subset S \) and local analytic coordinates \( x, y \)

on \( S \) near \( p \), i.e., \( x \) is the pull-back of a local coordinate on \( B \) (and \( y |_F \) is a local coordinate on \( F \))

locally at \( p \)

\[ \omega = F_1 \, dx + F_2 \, dy \]

\[ \omega' = F_3 \, dx + F_4 \, dy \]
$$\omega \cap \omega'^\prime = \left( F_1 \, dx + F_2 \, dy \right) \cap \left( F_3 \, dx + F_4 \, dy \right)$$

$$= \left( F_1 \, F_4 - F_2 \, F_3 \right) \, dx \wedge dy$$

$$S(\omega \cap \omega'^\prime) = \left( F_1 \, F_4 - F_2 \, F_3 \right) \, dy$$

$$H^0(\mathcal{A}) \rightarrow H^0(\mathcal{A}_S) \rightarrow H^0(\mathcal{A}_F) = H^0(\omega_F)$$

$$\omega \rightarrow \omega|_S \quad \omega|_F = F_2 \, dy$$

$$\omega' \rightarrow \omega'|_S \quad \omega'|_F = -F_4 \, dy$$

Claim: We can choose $\gamma$ s.t. $F_2(\gamma) = 0$, $F_4(\gamma) \neq 0$, $F_1(\gamma) \neq 0$

$F_2(\gamma) = 0$ means $\gamma \in \text{Zeros}(\omega|_F)$

$F_4(\gamma) \neq 0$ means $\gamma \notin \text{Zeros}(\omega|_F)$

$F_1(\gamma) \neq 0$ means $\gamma \notin \text{Zeros}(\omega|_S)$