1 Parallel Transport

Let $M$ be a $C^\infty$ manifold, $E$ a $C^\infty$ vector bundle on $M$, and $\nabla : C^\infty(E) \to C^\infty(E \otimes T^*_M)$ a connection. Let $\gamma : [0,1] \to M$ be a smooth curve in $M$. We may pull back the vector bundle $E$ to give $\gamma^*E$, a $C^\infty$ vector bundle on $[0,1]$ satisfying

$$(\gamma^*E)_t = E_{\gamma(t)}.$$ 

We may further pull back the connection $\nabla$ to give a connection on $\gamma^*E$;

$$\gamma^*\nabla : C^\infty(\gamma^*E) \to C^\infty(\gamma^*E \otimes T^*_M[0,1]).$$

Really, we have

$$\gamma^*\nabla : C^\infty(\gamma^*E) \to C^\infty(\gamma^*E \otimes \gamma^*T^*_M)$$

but following this by the projection $C^\infty(\gamma^*T^*_M) \to C^\infty(T^*[0,1])$ gives the desired pullback connection.

Let $U$ be a chart with local coordinates $x^1, \ldots, x^n$ and $s_1, \ldots, s_k$ a local basis of sections of $E$. Write

$$\gamma(t) = (x^1(t), \ldots, x^n(t))$$

and

$$\dot{\gamma}(t) = \frac{d}{dt}\gamma(t) = (\dot{x}^1(t), \ldots, \dot{x}^n(t)) = \sum_{i=1}^n \dot{x}^i(t) \frac{\partial}{\partial x^i}.$$ 

Define

$$\nabla \frac{\partial}{\partial x^i} s_l = \sum_{j=1}^k \Gamma^j_{i,l} s_j.$$ 

Then for any section $s = \sum a^i s_i$, we get a section of $E$ along $\gamma$ by putting $s(t) = s(\gamma(t))$. We may explicitly compute

$$\nabla_{\dot{\gamma}(t)} s(t) = \nabla \sum_{i=1}^n \dot{x}^i(t) \frac{\partial}{\partial x^i} s(t)$$

$$\quad = \sum_{i=1}^n \dot{x}^i(t) \nabla \frac{\partial}{\partial x^i} s(t)$$

$$\quad = \sum_{i=1}^n \dot{x}^i(t) \nabla \frac{\partial}{\partial x^i} \left[ \sum_{l=1}^k a^l(t) s_l(t) \right]$$

$$\quad = \sum_{i=1}^n \sum_{l=1}^k \dot{x}^i(t) \nabla \frac{\partial}{\partial x^i} a^l(t) s_l(t)$$

$$\quad = \sum_{i=1}^n \sum_{l=1}^k \dot{x}^i(t) \left[ \frac{\partial a^l(t)}{\partial x^i} \epsilon_l(t) + a^l(t) \sum_{j=1}^k \Gamma^j_{i,l} s_j(t) \right].$$

The above derivation is not entirely telling, but serves to provide the following proposition.
Proposition 1.1. Let $\gamma : [0,1] \to M$ be a smooth curve with $\gamma(0) = x$ and $\gamma(1) = y$. Then for any $e \in E_x = (\gamma^*E)_0$ there exists a unique smooth section $s$ of $\gamma^*E$ satisfying

1. $\nabla_{\dot{\gamma}(t)} s(t) = 0$ for any $t \in [0,1]$,

2. $s(0) = e$.

Proof. The calculation immediately preceding this Proposition shows that the vanishing of $\nabla_{\dot{\gamma}(t)} s(t)$ is equivalent to a system of linear differential equations on the coefficient functions $a$. The initial condition $s(0) = e$ provides an initial condition for these differential equations, and thus the Proposition follows from Picard-Lindelof. 

With the notation as in the previous Proposition, we define the parallel transport of $e$ along $\gamma$ as $P_\gamma(e) = s(1) \in E_y$.

It is a fact that $P_\gamma$ is a linear isomorphism $P_\gamma : E_x \to E_y$.

2 Holonomy Groups

Recall that a path $\gamma : [0,1] \to M$ is a loop if $\gamma(0) = \gamma(1)$. If $\gamma$ is a loop based at $x$, then parallel transport along $\gamma$ is an automorphism of $E_x$, i.e. $P_\gamma \in GL(E_x)$.

We define the holonomy group

$Hol_x(\nabla) := \{P_\gamma : \gamma$ a loop based at $x\}$.  

Holonomy groups have several nice properties.

1. $Hol_x(\nabla)$ is a Lie subgroup of $GL(E_x)$. (To prove that is a group, use the same types of arguments as with fundamental groups regarding concatenation/inversion of paths).

2. They depend only on connected components of $M$ in the following sense. Suppose $x$ and $y$ are in the same connected component as $M$. Pick a path $\gamma$ starting at $x$ and ending at $y$. Then parallel transport allows us to write $Hol_y(\nabla) = P_\gamma Hol_x(\nabla) P_\gamma^{-1}$.

Further, if we choose an abstract isomorphism $E_x \cong \mathbb{R}^k$, then in fact $Hol_x(\nabla)$ becomes a well defined subgroup of $GL_k(\mathbb{R})$ up to conjugation.

3. If $M$ is simply connected, then $Hol_x(\nabla)$ is connected. Indeed if $\gamma$ is a loop based at $x$ then it may be continuously shrunk to the constant path; the corresponding parallel transports will be a path connecting $P_\gamma$ to $Id$ in the holonomy group.
4. Let $hol_x(\nabla)$ denote the Lie algebra corresponding to $Hol_x(\nabla)$. Then $hol_x(\nabla)$ is a Lie subalgebra of $gl(E_x) = End(E_x)$. One can show that the curvature $R(\nabla)_x$ of $\nabla$ at $x$ lies in the linear subspace $hol_x(\nabla) \otimes \wedge^2 T^*_x M \subset End(E_x) \otimes \wedge^2 T^*_x M$. Thus holonomy places a linear restriction on curvature.

To expand on this point, recall $R(\nabla) : C^\infty(E) \to C^\infty(E \otimes T^*_x M)$. But now $Hom(C^\infty(E), C^\infty(E \otimes T^*_x M))$ is canonically isomorphic to $C^\infty(E^* \otimes E \otimes \wedge^2 T^*_x M) = C^\infty(End(E) \otimes \wedge^2 T^*_x M)$. Thus thinking of the curvature as a section of $End(E) \otimes \wedge^2 T^*_x M$, it can be shown that at a given $x$

$$R(\nabla)_x \in hol_x(\nabla).$$

Given a connection $\nabla$ on $T_M$, we can define connections on $T^{\otimes k}_M \otimes (T^*_M)^{\otimes l}$ for any $k,l$ as follows. First define a connection on $T^*_M$ by the rule

$$(\nabla_v \alpha)(w) = v(\alpha(w)) - \alpha(\nabla_v w).$$

One verifies that this is indeed a map $T^*_M \otimes T_M \to T^*_M$, giving a connection on $T^*_M$. Next, given any two vector bundles $E,F$ with connections $\nabla^E, \nabla^F$, we get a connection on $E \otimes F$ by

$$\nabla_v(e \otimes f) = \nabla^E_v e \otimes f + e \otimes \nabla^F_v f.$$ All this is to say, given a connection $\nabla$ on $T_M$ we can establish connections on $T^{\otimes k}_M \otimes (T^*_M)^{\otimes l}$ for any $k,l$. If $s$ is a $(k,l)$ tensor with $\nabla s = 0$, we say $s$ is a (covariantly) constant tensor.

We now have the following theorem, which says that the constant tensors are determined by holonomy.

**Theorem 2.1.** Let $M$ be a $C^\infty$ manifold, $x \in M$, $\nabla$ a connection on $T_M$, and $H = Hol_x(\nabla)$. Then $H$ acts naturally on $T^{\otimes k}_M \otimes (T^*_M)^{\otimes l}$. If $s \in C^\infty(T^{\otimes k}_M \otimes (T^*_M)^{\otimes l})$, then $\nabla s = 0$ implies that $s|_x$ is fixed by the $H$ action. Moreover, if $s|_x \in T^{\otimes k}_x \otimes (T^*_x)^{\otimes l}$ is fixed by the $H$ action, then $s|_x$ extends uniquely to a $(k,l)$ tensor $s$ with $\nabla s = 0$.

### 3 Riemannian Manifolds

A $C^\infty$ manifold $M$ is called **Riemannian** if it admits a **Riemannian metric** $g$. A Riemannian metric is a $(0,2)$ tensor $g$ which is symmetric, i.e.

$$g \in C^\infty(\text{Sym}^2 T^*_M),$$

and which defines a positive definite quadratic form on each fiber $T_x M$ for all $x \in M$. 

3.1 The Levi Civita Connection

Theorem 3.1 (Fundamental Theorem of Riemannian Geometry). Given a Riemannian manifold $M$ with metric $g$, there exists a unique torsion free connection $\nabla$ on $T_M$ such that $\nabla g = 0$.

The connection in the above Theorem is the Levi Civita connection. It can be defined explicitly by

$$2g(\nabla_u v, w) = u \cdot g(v, w) + v \cdot g(u, w) - w \cdot g(u, v) + g([u, v], w) - g([v, w], u) - g([u, w], v).$$

Thinking of connections as ways of taking derivatives, this says that the derivative of the Riemannian metric is zero with respect to the Levi Civita connection.

Next time, more on curvature of the Levi Civita connection.