Riemannian manifolds: A $C^\infty$ manifold is called Riemannian if it has a Riemannian metric $g$; this is a $(0,2)$-tensor $g \in C^\infty(T^*M \otimes T^*M)$ which is symmetric, i.e., $g \in C^\infty(\text{Sym}^2 T^*M)$ and is such that $g$ defines a positive definite quadratic form on $T^*_M x$ for all $x \in M$.

\[ g : \text{Sym}^2 T^*_M x \to \mathbb{R} \]
Parallel transport and holonomy groups:

\[ M \text{ C}^\infty \text{ manifold. } E \to M \text{ C}^\infty \text{ vector bundle. } \]

\[ \nabla : E \to E \otimes T^*_{M} \text{ a connection } \]

\[ \gamma : [0, 1] \to M \text{ a smooth curve. } \]

\[ \gamma^* E \text{ is a vector bundle on } [0, 1] \text{ with fiber } E_{\gamma(t)} \text{ at } t \in [0, 1]. \]

\[ \nabla \gamma \text{ defines a connection on } \gamma^* E: \]

\[ \gamma^* \nabla : \gamma^* E \to \gamma^* E \otimes T^*_{[0, 1]} \]

\[ \gamma^* \nabla : \gamma^* T_{\gamma(t)} \otimes \gamma^* E \to \gamma^* E \]

(This is the projection of \[ \gamma^* E \to \gamma^* E \otimes \gamma^* T^*_{M} \] to \[ T^*_{[0, 1]} \].)
in coordinates: \((x', \ldots, x^n)\) on \(M\)

\[
\gamma(t) = (x'(t), \ldots, x^n(t))
\]

\[
\dot{\gamma}(t) = (\dot{x}'(t), \ldots, \dot{x}^n(t)) = \sum_{i=1}^{n} \frac{\dot{x}^i(t)}{\partial x^i}
\]

\[
\nabla_{\dot{\gamma}(t)} e = \sum_{i=1}^{n} \frac{\dot{x}^i(t)}{\partial x^i} e = \sum_{i=1}^{n} \frac{\dot{x}^i(t)}{\partial x^i} e
\]

**Definition and Proposition:** Put \(\gamma(0) = x, \gamma(1) = y\).

Then \(\forall e \in E_x = (\dot{x}^i E)_0 \mathbb{F}\) smooth section \(s\) of \(\gamma^* E\) s.t. \(\gamma^* \nabla s = 0\), i.e., \(\nabla_{\dot{\gamma}(t)} s = 0 \quad \forall t \in [0, 1]\) and \(\gamma(0) = e\).

The parallel transport of \(e\) along \(\gamma\) to \(y\) is \(P_\gamma(e) := s(1) \in E_y = (\dot{y}^i E)_1\).

\(P_\gamma\) is a linear isomorphism \(P_\gamma : E_x \to E_y\).
Recall that \( \gamma \) is a loop if \( \gamma(0) = \gamma(1) = x \)

**Definition:** Suppose \( \gamma \) is a loop based at \( x \in M \), then \( P_\gamma \in \text{GL}(E_x) \). The holonomy group \( \text{Hol}_x(\gamma) \) at \( x \) is

\[
\text{Hol}_x(\gamma) := \{ P_\gamma : \gamma \text{ is a loop based at } x \}
\]

\( \subset \text{GL}(E_x) \)

The holonomy group has the following properties:

1. \( \text{Hol}_x(\gamma) \) is a Lie subgroup of \( \text{GL}(E_x) \)

\[
(\gamma \delta)(t) = \begin{cases} 
\delta(2t) & \text{if } t \in [0, \frac{1}{2}] \\
\gamma(1-2t) & \text{if } t \in [\frac{1}{2}, 1]
\end{cases}
\]

\[
\gamma^{-1}(t) = \gamma(1-t)
\]

Then \( P_{\gamma \delta} = P_\delta \circ P_\gamma \) and \( P_{\gamma^{-1}} = (P_\gamma)^{-1} \)
(2) $\text{Hol}_x (\triangledown)$ depends only on the connected component of $M$ to which $x$ belongs (up to "conjugation") if $x, y \in$ connected component of $M$.

Exist a path from $x$ to $y$.

Then $\text{Hol}_y (\triangledown) = P_y \text{Hol}_x (\triangledown) P_x^{-1}$

$\text{GL}(E_y)$

If $E_x \cong \mathbb{R}^k$, $E_y \cong \mathbb{R}^l$

$\text{Hol}_x (\triangledown) \circ \text{Hol}_y (\triangledown)$ is well-defined in $\text{GL}(k, \mathbb{R})$ up to conjugation.
(3) If $M$ is simply connected, then $\text{Hol}_x(\nabla)$ is connected: any path $\gamma$ can be shrunk to a point:

$$\gamma : \{0,1\} \times [0,1] \to M$$

$$\gamma_s : (0,1) \to M$$

$$\gamma_0(t) = x \quad \forall \ t \in [0,1]$$

Then $\gamma_s := \gamma_s$ is a path in $\text{Hol}_x(\nabla)$ from $\gamma_0$ to $\gamma_1 = \text{Id}$.

(4) Let $\text{hol}_x(\nabla) \subset \text{gl}(E_x) = \text{End}(E_x)$ be the Lie algebra of $\text{Hol}_x(\nabla)$. There one can show that the curvature $R(\nabla)_x \in \text{hol}_x(\nabla) \otimes \Lambda^2 T^* M$ recall $\nabla : E \to E \otimes \Lambda^2 T^*_M$. 
\[ R(\nabla) \in \mathcal{C}^0(\mathcal{E}^* \otimes \mathcal{E} \otimes \Lambda^q T^*_M) = \mathcal{C}^0(\text{End}(\mathcal{E}) \otimes \Lambda^q T^*_M) \]

at \( x \) \( \text{hol}_x(\nabla) \otimes \Lambda^q T^*_x M \subset \text{End}(\mathcal{E}_x) \otimes \Lambda^q T^*_x M \)

Assume \( M \) connected.

Theorem: Given a section \( e \in \mathcal{C}^0(\mathcal{E}) \)

\[ \nabla e = 0 \implies e(x) \text{ is fixed by } \text{hol}_x(\nabla) \]

If \( \nabla \) is a connection on \( T_M \), \( \nabla \) also defines connections on all tensor powers \( T^k_M \otimes T^*_M \).

A tensor \( S \) is called (covariantly) constant if \( \nabla S = 0 \).

Theorem: \( \nabla S = 0 \implies S(x) \text{ is fixed by } \text{hol}_x(\nabla) \).

\( (\text{P6} \ (S(x)) = S(y) \ \forall \delta \text{ from } x \to y) \)
Riemannian manifolds and the Levi–Civita connection

Fundamental theorem of Riemannian geometry:

There exists a torsion-free connection \( \nabla \) on \( T_m \) s.t. \( \nabla g = 0 \).
This is called the Levi–Civita connection.

You can define it via:

\[
2g(\nabla_v u, w) = u \cdot g(v, w) + v \cdot g(u, w) - w \cdot g(u, v) \\
+ g([u, v], w) - g([v, w], u) - g([u, w], v)
\]

The curvature of \( \nabla \) is a \((1, 3)\)-tensor

\[
\big( \bigwedge T_M \otimes T_M \otimes \bigwedge^2 T_M \big)^* \leftrightarrow T_M 

\rightarrow T_M \otimes \bigwedge^3 T_M
\]