In coordinates $(x^1, \ldots, x^n)\n$
\[ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \quad dx_1 \cdots dx_n \]
\[ R(V) = \sum R^a \frac{\partial}{\partial x^a} \otimes dx_1 \otimes dx_2 \otimes dx_d \]
\[ g = \sum g_{ab} \quad dx^a \otimes dx^b \quad g_{ab} = g_{ba} \]
"Compose" the curvature with $g$ to get a new expression:
\[ R : \quad R_{abcd} := \sum_{e=1}^{n} g^{ae} R^{e} \quad R_{abcd} \]
\[ R : T^*_M \rightarrow T^*_M \otimes \Lambda^2 T^*_M \otimes \Lambda^2 T^*_M \rightarrow \widetilde{R} \rightarrow T^*_M \otimes \Lambda^2 T^*_M \]
\[ \widetilde{R} \in C^\infty (T^*_M \otimes \Lambda^2 T^*_M) \quad \text{a mini} \]
This has some nice properties.
(1) \( \tilde{R}_{abcd} = - \tilde{R}_{bacd} = - \tilde{R}_{abdc} = \tilde{R}_{cdab} \)
This means \( \tilde{R} \in C^\infty (\text{Sym}^2 \Lambda^2 T^* M) \)

(2) First Bianchi identities:
\[ \tilde{R}_{abcd} + \tilde{R}_{acdb} + \tilde{R}_{adbc} = 0 \]

(3) Second Bianchi identities:
\[ \frac{\partial}{\partial x^c} \tilde{R}_{abcd} + \frac{\partial}{\partial x^c} \tilde{R}_{abdc} + \frac{\partial}{\partial x^c} \tilde{R}_{adbc} = 0 \]

Definition: The Ricci curvature tensor of \( g \) is a \((0, 2)\) tensor, obtained by "contracting" \( \tilde{R}(\nabla) \). Its components are \( R_{ab} := \sum_c R^c_{ac} \), which satisfies \( R_{ab} = R_{ba} \).
We say $g$ is Einstein if the Ricci curvature is a constant multiple of the metric. We say $g$ is Ricci-flat if the Ricci curvature is 0.

**Riemannian holonomy:**

$(M, g)$ Riemannian manifold, $\nabla = \text{Levi-Civita}, x \in M$

$\text{Hol}_x (g) := \text{Hol}_x (\nabla)$  

$\text{hol}_x (g) \subseteq \text{End} (T_M) = T_M \otimes T_M^* \xrightarrow{g} (T_M)^* \otimes (T_M)^* \otimes \ldots$

compose with $g : \{T^a_b\} \in T_M \otimes T_M^* \xrightarrow{g} (T_M)^* \otimes (T_M)^* \otimes \ldots$

$\{T^a_c\} : T_M \rightarrow T_M \xrightarrow{g} T_M^*$

$\text{Tab} := \sum_c g_{ac} T^c_b$
Via the identification $T_x M \otimes T_x^* M \cong \mathfrak{g}$, we have

\[ \text{hol}_x(g) \subset \Lambda^2 T^*_x M \]

Claim:

Then $\sum_{a b c d} \in S^2 \text{hol}_x(g) \subset S^2 \Lambda^2 T^*_x M$ \hspace{1cm} \forall x \in M.

Some notions we need for Berger's classification of Riemannian holonomy groups:

Product: If $(M, g)$, $(N, h)$ are Riemannian of dim $m$ and $n$ respectively, then we can form $(M \times N, g \times h)$ of dim. $m + n$:

- at each $(x, y) \in M \times N$, $(g \times h)(x, y) = gx + ty$

\[ T_{M \times N} = T^* M \otimes T^* N \]
\[ g \times h := T^*_M g_{\mu} + T^*_N g_{\nu} \in S^2(T^*_M N) \oplus S^2(T^*_N M) \]

**Definition:** A Riemannian manifold is called (locally) reducible if every point has a neighborhood isometric to a product. It is called irreducible if it is not locally reducible.

**Prop.:** \( \text{Hol}(g \times h) = \text{Hol}(g) \times \text{Hol}(h) \).

**Theorem:** If \((M, g)\) is irreducible at \(x\), then \( \mathbb{R} \cong T_x M \) is an irreducible representation of \( \text{Hol}_x(g) \).
Symmetric spaces: (basically Riemannian manifolds that are homogeneous spaces.)

Definition: A Riemannian manifold is called symmetric if \( \forall f \in M, \exists \sigma \text{ an isometry } \sigma : M \to M \)
\( n.t. \) \( \sigma^2 = \text{Id}_M \) and \( f \) is an isolated fixed point of \( \sigma \).

Definition: A geodesic is a parameterized curve
\( \gamma : (a, b) \to M \) n.t.: \( \nabla_{\gamma'} \dot{\gamma}(t) = 0 \) \( \forall t \in (a, b) \).

Theorem: \( \forall p \in M, \forall v \in T_p M, \exists ! \) geodesic
\( \gamma : (a, b) \to M \) n.t.: \( \gamma(0) = p \), \( \dot{\gamma}(0) = v \).

Definition: \((M, g)\) is complete if every geodesic can be defined on \( \mathbb{R} = (a, b) \).
Proposition: Let $(M, g)$ be a connected, simply-connected symmetric space. Then $(M, g)$ is complete. Put $G := \{\phi \circ \phi^{-1} | \phi, \phi \in M\} \subset \text{Isom}(M)$. Then $G$ is a connected Lie group acting on $M$.

Choose $p \in M$. Put $H := \text{Stab}(p) \subset G$.

Then $H$ is a closed, connected Lie subgroup of $G$ and $G/H \rightarrow M$ is a diffeomorphism. 

$g \mapsto g \cdot p$