First series of examples: Hilbert schemes of points.

$S$: a compact complex manifold of dim. $2$

$S^r$: the $r$-th Cartesian power of $S$

$S^{(r)} := S^r / S^r_n$ the $r$-th symmetric power of $S$

The action of $S^r_n$ is not free everywhere.

$\Delta$: the diagonal of $S^r$ where the $i$-th and $j$-th components are equal.

A general point $(x_1, \ldots, x_i, x_i, x_{i+1}, \ldots, x_j, \ldots, x_n) \in \Delta$

has stabilizer $(ij) \in S_n$

$\text{codim } \Delta_j = 2$ in $S^r$. 

\[ \pi: S^r \to S^{(r)} \] is the quotient morphism.

By the purity of the ramification loci of a morphism of smooth varieties, \( S^{(r)} \) is singular along \( \pi(U \Delta_{ij}) =: D \). The diagonal of \( S^{(r)} \) is irreducible.

We desingularize \( S^{(r)} \): \( S^{[r]} := \) the Hilbert scheme of length \( r \) artinian subschemes of \( S \) \\
\( \varepsilon: S^{[r]} \to S^{(r)} \) \\
\( Z \to \) underlying cycle of \( Z \)
For any distinct points $x_1, \ldots, x_n$ in an Artinian subscheme of $X$ of length $n$ supported on $x_1, \ldots, x_n$ and the cycle associated with it is $x_1 + \cdots + x_n$.

So, $\epsilon : S^{(n)} \setminus \epsilon^{-1}(D) \to S^{(n)} \setminus D$ is an isomorphism.

Let $D^\times$ be the open subset of $D$ of points where exactly 2 coordinates are equal.

Given $2x_1 + x_2 + \cdots + x_{n-1} \in D^\times$, to specify an Artinian subscheme of length $n$ with cycle $2x_1 + \cdots + x_n$, we need to specify a tangent line to $S$ at $x_1$. 
So the set ofartinian subschemes with cycle 
\[2x_1 + \ldots + x_n\] is naturally identified with \( \mathbb{P}^{T_1} S \).

Let \( \mathcal{S}^{(n)}_* \) be an open subset of points where at 
most two of the coordinates coincide.

Let \( \mathcal{S}^{(n)}_* \) be the inverse image of \( \mathcal{S}^{(n)}_* \).

The fibers of the map \( E: \mathcal{S}^{(n)}_* \to \mathcal{S}^{(n)}_* \)
along \( D_* \) are naturally identified with \( \mathbb{P}^{T_1} x \in S \).

One can prove:

**Theorem 1:** \((\mathcal{S}^{(n)}_*, D_*)\) is locally isomorphic

(as a complex analytic space) to
(2) \( S^{(n)}_x = \text{the blow up of } S^{(n)}_x \text{ along } D_x \)

(3) Also, if \( \text{Bl}_\Delta (S^*_x) \) is the blow up of \( S^*_x \) along the union of the diagonals, then the action of \( G^* \) lifts to \( \text{Bl}_\Delta (S^*_x) \), and

\[
S^{[n]}_x = \frac{\text{Bl}_\Delta (S^*_x)}{G^*}
\]

So we have the pull-back diagram:

\[
\begin{array}{ccc}
\text{Bl}_\Delta (S^*_x) & \xrightarrow{\eta} & S^{(n)}_x \\
\downarrow & & \downarrow \\
S^{[n]}_x & \xrightarrow{\varepsilon} & S^{(n)}_x
\end{array}
\]
Given a holomorphic form $\omega$ on $S$, we obtain holomorphic forms $\pi_i^* \omega$ on $S_i$.

$\psi := \pi_1^* \omega + \cdots + \pi_n^* \omega$ is invariant under the action of $G_n$. The pull-back of $\psi$ to $\text{Bl}_A(S^n)$ is also invariant under the action of $G_n$, so $\exists \varphi$ on $S^{[n]}$ s.t. $\varphi^* \psi = \varphi^* \varphi$.

Proposition: If $K_S$ is trivial, then $S^{[n]}$ admits a holomorphic symplectic form.

Proof: Let $\omega$ be a generator of $K_S$; then we have $\varphi$ and $\varphi$ as above. We need to show that $\varphi$ extends to $S^{[n]}$ as an everywhere non-degenerate form.
This means $\nu^* \varphi$ doesn't vanish anywhere.

$\nu$ extends to all of $S^\infty$ because $S^\infty \setminus S^\infty_*$ has codim. $\geq 2$.

$\nu^* \varphi$ is a section of $K_{S^\infty}$, so the locus where it vanishes is a canonical divisor.

The morphism $\text{Bl}_\Delta (S^\infty_*) \longrightarrow S^\infty_*$ is ramified along the exceptional divisors $E_{ij} = \text{inverse image of } \Delta_{ij}$.

$$K_{\text{Bl}_\Delta (S^\infty_*)} = \nu^* K_{S^\infty} + \sum_{i < j} E_{ij}.$$  

$$\text{div.} (\nu^* \Lambda^\infty \varphi) = \nu^* \text{div} \Lambda^\infty \varphi + \sum_{i < j} E_{ij}.$$  

However $$\text{div} (\nu^* \Lambda^\infty \varphi) = \text{div} (\eta^* \Lambda^\infty \varphi) = \sum_{i < j} E_{ij}.$$
because: $\nabla \eta^* \psi = \eta^* \nabla \psi = \eta^* \mathbf{L}$

so $\text{div} \left( \rho^* \nabla \psi \right) = 0 \Rightarrow \text{div} \nabla \psi = 0 \quad Q.E.D.$