\[ X := \frac{L^* \cap \Delta^*}{L^*} \subset \mathbb{P}^5 \]

Pfaffian cubic fourfold in \( \mathbb{P}^5 \).

**Prop. 5:**

(i) \( F(X) \cong S^2 \)

(ii) \( X \) is rational.

**Proof:**

(ii) \( Z \subset \mathbb{P}(V) \times X \)

\[ Z := \{ (v, \varphi) \mid v \in \ker \varphi \} \]

\[ Z \xrightarrow{\mu_2} X \text{ is surjective} \]

All forms \( \omega \) on \( X \) have rank 4 \( \Rightarrow \) fibers of \( \mu_2 \) are \( \mathbb{P}^1 \)s.

i.e., \( \mu_2 \) is a \( \mathbb{P}^1 \)-bundle on \( X \). (Fiber at \( \varphi \) = \( \mathbb{P}(\ker \varphi) \).

\[ Z \xrightarrow{\mu_1} \mathbb{P}(V) \text{ is birational because:} \]

generic fibers are points because:
$Z \ni (v, y) \xrightarrow{\mu_1} \varnothing \in \mathcal{P}(V)$

$\mu_1^* [v] = \{ \varnothing \mid v \in \ker \varnothing \}$

Choose a basis so that $v = (1, 0, 0, 0, 0, 0, 0) \in V$

$\forall \varnothing \in \ker \varnothing \implies \varnothing \rightarrow \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$

$Z \subset \mathcal{P}(V) \times \Delta^* \rightarrow \Delta^*$

$\int \mu_1 \text{ dim } \tilde{Z} = 14$

$\mathcal{P}(V) \text{ dim } \mu_1^* (v) = 9$

$\Rightarrow \mathcal{P}_9$

So all forms in $\mu_1^* (v)$ can be represented by a matrix as above.

Now cut with $L^+ : 9$ generic linear conditions.
\[ \text{Therefore, } L^+ \text{ is a point for } L^+ \text{ generic.} \]

\[ \text{So, } \mu_1^*(r) = \{ pt \} \text{ for } r \text{ generic and } L^+ \text{ generic.} \]

\[ \text{So, } \mu_1 : Z \to \text{IP}(V) \text{ is birational for } L^+ \text{ generic.} \]

\[ \therefore \quad \begin{array}{ccc}
Z & \xrightarrow{\text{IP}^1\text{-bundle}} & X \\
\downarrow \text{birational} & & \downarrow \text{birationally to } X \\
\text{IP}(V) & & \text{The proper transform for } u^H \text{ of a generic hyperplane in } \text{IP}(V) \text{ maps birationally to } X: \]

\[ \text{because } \mu_1^*(y) \cap H \leftrightarrow r \in \ker(\cup H) \text{ one point for } y \text{ and } H \text{ generic.} \]
Fin (i) we write inverse isomorphisms between $F$ and $S^{[2]}$.

(a) $S^{[2]} \rightarrow F$ Choose $P \neq Q \in S$ then $P$ and $Q$ are two distinct 2 dim. subvector spaces of $V$. Claim: for $L$ generic $P+Q \subseteq V$ has dimension 4.

$\varphi \in X = \Delta \cap L^\perp \quad S = S \cap L$

means $\varphi \mid_P \equiv 0 \quad \forall P \in S$

$$\left( \forall P = \langle v_1, v_2 \rangle \quad : \quad \varphi(v_1, v_2) = 0 \right)$$

So $\varphi \mid_{P+Q}$ depends on 3 projective parameters.

So $\exists$ 1 parameter family of $\varphi \in X \setminus S$. 
\[ \gamma \mid_{P + \mathbb{Q}} = 0 : \text{this is a line} \ d(P + \mathbb{Q}) \in F. \]

On the exceptional divisor, we have \( P \in S \) with a tangent direction \( w \in T_p S \subset T_p G = \text{Hom}(\mathbb{P}^1, V_p) \).

The image of \( w \) in \( V_p \) has dimension 2, the inverse image of this in \( V \) is a 4-dimensional subspace \( \langle e \rangle \) containing \( P \). The set of \( \gamma \in X \) s.t. \( \gamma \mid_{\langle e \rangle} = 0 \) is again a line in \( X \).