CURVES ON GENERIC KUMMER VARIETIES

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Introduction. In this paper we deal with curves of small geometric genus on Kummer varieties. In section 1 we prove a rigidity theorem (see theorem 1). Let $C$ be a curve of genus $g$ lying in the Kummer variety of a $q$-dimensional Abelian variety. Assuming that the Abelian variety is generated by the inverse image of the curve, theorem 1 states that we have rigidity if $g < q - 1$. The prototype of this result is the fact that a Kummer surface has a global holomorphic $(2, 0)$ form, so it cannot be covered by rational curves. The proof relies on an elementary, but very interesting, lemma of Xiao (cf. [4]). In section 2 we prove a nonexistence theorem in the hypothesis of generality of the Kummer variety for $g < q - 2$. Here we degenerate to Kummer varieties of nonsimple Abelian varieties and use theorem 1. Section 2 can be seen as a method of transforming a rigidity theorem into a nonexistence one. The most surprising consequence is the fact that a generic Abelian variety of dimension $\geq 3$ does not contain hyperelliptic curves of any genus. In section 3 we give some examples. We work over the field of complex numbers.

Section 1. Let $A$ be an Abelian variety of dimension $q > 1$, $K = K(A)$ the Kummer variety of $A$, and let $\pi: A \rightarrow K(A)$ be the quotient map. Let $C$ be a smooth curve of genus $g$, and $\varphi: C \rightarrow K$ a nonconstant morphism. We assume that $\pi^{-1}(\varphi(C))$ generates $A$ as a group (this is automatic if $A$ is a simple Abelian variety).

We will say that $(C, \varphi)$ is rigid if the image in $K$ of any deformation of $(C, \varphi)$ is contained in $\varphi(C)$. If $\varphi$ is birational onto its image this means that $\varphi(C)$ cannot be deformed in $K$ as curve of geometric genus $g$.

Theorem 1. If $g < q - 1$, $(C, \varphi)$ is rigid in $K$.

Proof. If $(C, \varphi)$ is not rigid there exist data $(S, B, p, b, \sigma)$ where:

$-$ $S$ is a smooth analytic surface,
$-$ $B$ is a smooth analytic curve,
$-$ $p: S \rightarrow B$ is a proper smooth morphism,
$-$ $b$ is a point of $B$ such that $p^{-1}(b) \cong C$,
$-$ $\sigma: S \rightarrow K$ is a map whose restriction to $p^{-1}(b)$ is the map $\varphi$, and such that the image of $\sigma$ has dimension 2.

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We let $S'$ be the normalization of the fiber product of $\sigma$ and $\pi$:

$$
\begin{array}{ccc}
S' & \xrightarrow{\sigma'} & A \\
\downarrow j & & \downarrow \pi \\
S & \xrightarrow{\sigma} & K
\end{array}
$$

(1)

$S'$ is a double covering of $S$ with an involution $i$ which preserves the fibers and is induced by $-1_A$ on the image of $\sigma'$. We let $p': S' \to B$ be the composition $p \circ j$. We can suppose that $C$ is the generic fiber of $p$, and we let $C'$ be the corresponding fiber of $p'$ ($C' = j^{-1}(C)$). From the facts that $\pi^{-1}(\varphi(C))$ generates $A$ as a group and that $g < q - 1$, it follows that $C'$ is irreducible, and then that $S'$ is an irreducible surface. Moreover by the normality of $S'$ we get that $S'$ is smooth near $C'$. We now consider the linear map $\delta: \Lambda^2 H^{1,0}(A) \cong H^{2,0}(A) \to H^{1,0}(C')$ obtained by composing the map $p^*: H^{2,0}(A) \to H^{2,0}(S')$ with the restriction map $r: H^{2,0}(S') \to H^0(C', \omega_{C'})$, where $\omega_{C'}$ is the canonical sheaf on $C'$; notice that $r$ is defined when a trivialization of $\omega_{C'}$ is chosen. We notice that $-1_A$ acts trivially on $\Lambda^2 H^{1,0}(A)$, so the image of $\delta$ is invariant under the action induced by $j$; therefore $\text{Im}(\delta) \subseteq H^0(C', \omega_{C'})^{(\mathcal{O}_C)} \cong H^0(C, \omega_C)$. In particular, $\dim(\text{Im}(\delta)) \leq g < q - 1$. Theorem 1 is implied by the following lemma due to Xiao.

**Lemma 1** (cf. [4]). Let $M$ be an irreducible surface, $D$ a smooth analytic curve, $\pi: M \to D$ a smooth proper flat map with generic fiber $F$. Let $X$ be an Abelian variety of dimension $q$ and $\Psi: M \to X$ a morphism. Let $\delta$ be the linear map $H^{2,0}(X) \to H^{1,0}(F)$ described above.

Suppose that: i) $\Psi(M)$ has dimension 2, and ii) $\Psi(F)$ spans $X$ as a group. Then $\dim(\text{Im}(\delta)) \geq q - 1$.

**Proof.** In [4] the lemma is stated for the special case in which $D$ is $\mathbb{P}^1$ and $X$ is the Albanese variety of $M$, but its proof works without any change in our case. □

**Remark 1.** For $g = 0$, theorem 1 is equivalent to a rigidity theorem for hyperelliptic curves on an Abelian variety. In fact, let $Y$ be a hyperelliptic curve with involution $i$; let also $p \in Y$ be a Weierstrass point (i.e., $i(p) = p$) and let $f$ be a non-constant map $f: Y \to A$. Suppose $f(p) = O_A$. Then there exists a canonical factorization of $f$:

$$
Y \xrightarrow{\text{Ab}} J(Y) \xrightarrow{\gamma} A \quad f = \gamma \circ \text{Ab}
$$

where $\text{Ab}$ is the Abel map with base point $p$ and $\gamma$ is a group morphism. By Abel’s theorem $-1_{J(Y)}$ acts on the image of the Abel map as the hyperelliptic involution $i$. It follows that $-1_A$ acts on the image of $f$ as an involution and then its image in $K(A)$ is a rational curve. (From this it follows quickly that the nonrational images
of hyperelliptic curves are again hyperelliptic curves, provided elliptic curves are considered as special cases of hyperelliptic curves). So the only hyperelliptic deformations of a hyperelliptic curve in a fixed Abelian variety are just the translations.

Suppose $\Psi: M \to D$ is a hyperelliptic fibration of a surface over a curve (i.e., all the fibers are hyperelliptic curves) such that the irregularity of $M$, $q(M)$, is bigger than the genus of $D$, $g(D)$. In this case the Albanese variety of $M$ is isogenous to a product of the Jacobian of $D$ and an Abelian variety $A$ of dimension $q(M) - g(D)$. Taking a (possibly ramified) covering $D' \to D$ it is possible to suppose that the induced fibration $\Psi': M' \to D'$ has a section of Weierstrass points. Using this section it is possible to construct a map $M' \to A$ that sends the section to the zero of $A$. From theorem 1 it follows that there exists a hyperelliptic curve $E$ of genus $q(M) - g(D)$ such that there is a morphism (of surfaces fibered over $D'$) from $M'$ to $D' \times E$. Similar considerations apply to bielliptic curves in Abelian varieties of dimension $\geq 3$, etc.

Another application is that a linear system on an Abelian surface contains only a finite number of hyperelliptic curves. We notice that in $M_g$ the hyperelliptic locus has dimension $2g - 1$, whereas a complete linear system over an Abelian surface has dimension $g - 2$. So in the moduli variety $M_g$ the intersection between the hyperelliptic locus and the image of any complete linear system has the right dimension.

Section 2. The aim of this section is to prove the following:

**Theorem 2.** Let $K$ be the Kummer variety of a generic principally polarized Abelian variety (p.p.a.v.) $A$ of dimension $q \geq 3$, (here generic means that $A$ represents "a point outside a countable union of proper analytic subvarieties" in its moduli space). Then $K$ does not contain any curve of geometric genus $< q - 2$.

**Remark.** As will be clear from the proof, the theorem holds for any fixed polarization.

In the sequel we use the following facts:

**Fact 1.** Let $H_q$ be the Siegel space that parametrizes isomorphism classes of Abelian varieties of dimension $q$ together with the choice of a symplectic basis of the lattice. Let $A$ be any p.p.a.v.. Then the subset $I(A)$ of $H_q$ whose points represent all the Abelian varieties isogenous to $A$ is (analytically) dense in $H_q$.

**Fact 2.** Let $M_g$ be the moduli space of curves of genus $g$, with $g \geq 3$. The subset $R$ of $M_g$ parametrizing all the curves having a nontrivial map onto some curve of genus $\geq 2$ is a closed subscheme of $M_g$ (in other words it is a finite union of closed subvarieties). Moreover the scheme $R'$ parametrizing all the triples $(C, f, D)$, where $C \in R$, $D$ is a smooth curve of genus bigger than 1, and $f: C \to D$ is a nontrivial morphism, has a finite number of irreducible components and, by the de Franchis-Severi theorem (cf. [1], [2]) the map $k: R' \to R$ defined by $k(C, f, D) = C$ is finite-to-one.
All these results are well-known, except possibly for the fact that the de Franchis locus $R$ is closed in $M_g$; this will be proved in the Appendix.

Using standard arguments, theorem 2 is implied by the following:

**Proposition.** Consider the diagram

$$
\begin{array}{ccc}
\pi: \mathcal{A} & \longrightarrow & \mathcal{K} \\
\downarrow & & \downarrow \\
U & & \\
\end{array}
$$

where $U$ is an open subset of $H_q$, $q \geq 3$, and

- $\mathcal{A}$ = the tautological bundle of Abelian varieties over $U$,
- $\mathcal{K}$ = the associated Kummer bundle.

Let $p: \mathcal{C} \rightarrow U$ be a smooth proper family of irreducible curves of genus $g$. Suppose that $\sigma: \mathcal{C} \rightarrow \mathcal{K}$ is a nontrivial map of families. Then $g \geq q - 2$.

**Proof.** The proof is by contradiction. Suppose $g < q - 2$. We can suppose $U$ small enough. First of all, as in section 1 we can take the normalization of the fibered product of $\pi$ and $\sigma$, so we obtain a smooth connected family of curves of genus $p$, $\mathcal{C}' \rightarrow U$, together with a map $\sigma': \mathcal{C}' \rightarrow \mathcal{A}$. Since the generic Abelian variety is simple we can assume that, for every point $t \in U$, the image of the map

$$\sigma_t: C_t \rightarrow A_t$$

generates $A_t$ as a group.

Let $B$ be any p.p.a.v. of dimension $q - 1$, and let $U_B$ be the subset of $U$ defined by

$$U_B = \{ [A] \in U: A \text{ is isogenous to } B \times E, \text{ where } E \text{ is an elliptic curve} \}.$$

By Fact 1, $U_B$ is dense in $U$, and moreover $U_B$ is a countable union of analytic curves obtained by fixing the topological type of the isogeny and moving the elliptic curve. We let $\Delta \subset U_B$ be an analytic disk and we restrict all our families $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{A}$ and $\mathcal{K}$ to $\Delta$. By construction there exists a projection map $s: \mathcal{A}|_{\Delta} \rightarrow B \times \Delta$. By taking the obvious composition maps we then obtain a commutative diagram of families over $\Delta$:

$$
\begin{array}{ccc}
\mathcal{C}' & \longrightarrow & B \times \Delta \\
\downarrow & & \downarrow \\
\mathcal{C} & \longrightarrow & K(B) \times \Delta \\
\end{array}
$$
where $K(B)$ is the Kummer variety of $B$. By theorem 1 we get that the image of $\sigma$ is a curve. We let $D$ be the normalization of the image of $\sigma'$. The family $\mathscr{C}'$ is not trivial because the fiber curves of genus $p$ admit maps onto elliptic curves with varying moduli. Then it follows that the genus of $D$, $g(D)$, must be less than $p$. On the other hand, $D$ must generate $B$ as group ($D$ is image of $C'_i$, which generates $A_i$), so $p > g(D) \geq \dim(B) \geq 2$. We denote by $\ell_i$ the natural map

$$\ell_i: C'_i \to D.$$ 

Let $Z$ be the countable set of all curves $D$ obtained by considering all possible analytic disks in $U_B$.

It follows from Fact 1 and Fact 2 that the map $\eta: U \to M_p$ induced by $\mathscr{C}'$ sends $U$ to $R$ (in fact $\eta: U_B \to R$, $\overline{U}_B = U$ and $R$ is closed in $M_p$). By taking a smaller $U$ we can suppose that $\kappa: \kappa^{-1}(\eta(U)) \to \eta(U)$ is a finite trivial covering; that is

$$\kappa^{-1}(\eta(U)) = \bigcup \{ V_i | i = 1, \ldots, N \},$$ 

$$V_i \cap V_j = \emptyset \text{ if } i \neq j,$$

$$\kappa: V_i \to \eta(U)$$

is a homeomorphism.

Then we have a map $L: U_B \to \bigcup V_i$ defined by $L(t) = (C'_i, \ell_i, D_i)$. Let $U_{B_i}$ be the inverse image of $V_i$: $U_{B_i} = L^{-1}(V_i)$, and let $Z_i$ be the corresponding subset of $Z$. For some $i$, the closure of $U_{B_i}$ must contain an open set.

So by taking a smaller $U$, we can fix a lifting of $\eta, L: U \to R'$, such that the curves $D_i$ are in the closure of the image in $M_n$ of the set of all the curves of genus $n$ contained in $Z$. So there is a family of smooth curves $\mathcal{D} \to U$ together with a morphism $L: \mathcal{C}' \to \mathcal{D}$ whose fibers $L_i$, when restricted to some disk $\Delta$ in $U_B$, are the $\ell_i$.

To end the proof of the proposition we take the Jacobian bundles over $U$ of $\mathcal{C}'$ and $\mathcal{D}$, $J(\mathcal{C}')$ and $J(\mathcal{D})$, respectively. From the above maps we get the following morphism between fibrations of algebraic groups:

$$\sigma'^*: \mathcal{A} \to J(\mathcal{C}'),$$

$$L^*: J(\mathcal{D}) \to J(\mathcal{C}').$$

By taking $U$ simply connected we get that, topologically, all our bundles are products of $U$ by tori of real dimensions $2q, 2p, \text{and } 2n$, respectively. We notice that, when the bases of the lattices are fixed, $\sigma'^*$ and $L^*$ are represented by fixed matrices with integral entries.

This allows to conclude that, for any $t \in U$, the dimension of

$$\sigma'^*(A_i) \cap \ell_i^*(J(D_i)) \subset J(C'_i)$$

does not depend on $t$. 


For general $t$, $A_t$ is a simple Abelian variety, so we have only two possibilities:

i) $\ell^*(J(D_t)) \supseteq \sigma^*_t(A_t)$,

ii) $\dim(\sigma^*_t(A_t) \cap \ell^*(J(D_t))) = 0$ for any $t \in U$.

But, by construction, for some $t \in \Delta \subset U_d$ we must have a commutative diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{s^*_t} & A_t \\
\downarrow & & \downarrow \\
J(D_t) & \xrightarrow{\ell^*_t} & J(C_t)
\end{array}
\]

where the maps $J(D_t) \to J(C_t)$ and $A_t \to J(C_t)$ are $\ell^*_t$ and $\sigma^*_t$, respectively, and $s^*_t$ the dual of the projection $s_t: A_t \to B$. So $\ell^*_t(J(D_t)) \supseteq \sigma^*_t(s^*_t(B))$, and then ii) is impossible. On the other hand if i) is true we should have that $\ell^*_t(J(D_t))$ for any $t \in \Delta$, but $D_t = D$ does not depend on $t$, so $J(D)$ should contain infinitely many elliptic curves with varying moduli, which is absurd.

Section 3.

Example 1. Let $C$ be a bielliptic curve of genus 4, $f: C \to E$, where $E$ is an elliptic curve, $\deg(f) = 2$, and $J(C) \to E$ is the induced map. Set $A = J(C)/f^*(E)$. Taking the projection $q: C \to A$, we get a bielliptic curve in an Abelian threefold and then a nonconstant map of $E$ in its Kummer variety $K(A)$. Notice that $q(C)$ generates $A$ as a group and then $q$ is birational onto the image. Moreover we can endow $A$ with a polarization by considering the image of the obvious map from the two-fold symmetric product $C(2)$ to $A$ induced by $q$. We have a natural map $\Phi$ from the Hurwitz scheme of bielliptic curves to a suitable space of moduli of polarized Abelian varieties. Notice that both spaces have dimension 6. Then by the rigidity theorem (see theorem 1) the bielliptic deformations of $q(C)$ are just the translations. This proves that $\Phi$ is finite-to-one. The generic Abelian variety of dimension three then contains a bielliptic curve and finally the generic Kummer of dimension three contains an elliptic curve. In the same way it is possible to see that the generic Kummer variety of dimension three contains families of curves of genus 2 (take double coverings of curves of genus 2 with curves of genus 5).

Example 2. Let $K$ be the Kummer variety of a Jacobian $J$ of a hyperelliptic curve $C$ of genus $g$. $J$ is the Prym variety (see [3]) of a two-dimensional family of hyperelliptic curves of genus $g + 1$. It follows that $K$ contains a two-dimensional family of curves of genus $g + 1$.

The previous examples prove that theorem 1 is almost sharp, and theorem 2 is sharp in dimension 3. (Consider also the “trivial” example of the Kummer varieties of Jacobian varieties).

Remark. Theorem 2 proves (see remark 1 of section 1) that a generic Abelian variety $A$ of dimension $\geqslant 3$ does not contain any hyperelliptic curve. This is then equivalent to the following:
**Corollary.** It is not possible to represent a generic Abelian variety of dimension $\geq 3$ as a quotient of a hyperelliptic Jacobian.

**Problem.** Does a generic Abelian variety of dimension $q \geq 4$ contain any trigonal curves?

**Appendix.** We will prove that $R \subset M_g$, the locus of curves of genus $g$ that admit a map onto a curve of genus $g'$, with $g > g' \geq 2$, is closed with respect to the natural topology. Let then $[C]$ be a point of the closure of $R$: we shall prove that it belongs to $R$.

We first fix an open neighborhood of $[C]$, $U$. Possibly after a change of base, we may suppose that there exist a universal family $\pi: C \to U$ and a point $0$ of $U$ such that $\pi^{-1}(0) \cong C$. By shrinking $U$ if necessary we may assume that $U$ is contractible, and that $\pi$ has a section $p$. Over $U$ we have:

i) the Jacobian fibration of $\pi$, $J(C) \to U$

ii) the Cartesian product fibration: $C \times U \to U$.

There is a natural "difference" morphism of fibrations

$$e: C \times U \to J(C)$$

$$e_t(x, y) = \mathcal{O}(x - y) \text{ for } (x, y) \in C_t \times C_t.$$  

Using the section $p$ we also get an Abel-Jacobi morphism:

$$a: C \to J(C)$$

$$a_t(x) = \mathcal{O}(x - p(t)) \text{ for } x \in C_t.$$  

By assumption, there is a sequence $\{u_n\}_{n \in \mathbb{N}}$ of points of $U$ converging to $0$ such that each $C_n = \pi^{-1}(u_n)$ belongs to $R$. In other words, for any $n$ there exists a nonconstant map $f_n: C_n \to D_n$ with $g(D_n) \geq 2$ and $g > g(D_n)$. Consider the effective reduced divisors of $C_n \times C_n$: $T_n = \{(x, y) \in C_n \times C_n: f_n(x) = f_n(y)\}$. From Hurwitz' formula we get that $T_n$ has bounded degree with respect to the hyperplane $H_n = C_n \times \{p_n\} + \{p_n\} \times C_n$, where $p_n = p(u_n)$. So there must exist some flat limit of $T_n's$, $T \subset C \times U C$. After passing to a subsequence, we may assume that $\lim_{n \to +\infty} T_n = T \subset C \times C$, and that $g(D_n)$ and $\deg(f_n)$ do not depend on $n$. Set $g' = g(D_n)$, and $d = \deg(f_n)$.

In this way also the degree $b$ of the ramification divisors of the $f_n$ does not depend on $n$. So $0 \leq b = (T_n - \Delta_n) \cdot \Delta_n = (T - \Delta) \cdot \Delta$, where $\Delta_n$ denotes the diagonal of $C_n \times C_n$ and $\Delta$ the diagonal of $C \times C$. Since $g \geq 2$, the self-intersection of $\Delta$ is negative, so it follows at once that $T \neq k\Delta$, that is, there exists a component of $T$ different from $\Delta$. Setting

$$t_n = c_1(T_n - dH_n) \in H^2(C_n \times C_n, \mathbb{Z}),$$
\( \tau_n \) corresponds via the Kunneth isomorphism, to an element in \( H^1(C_n, \mathbb{Z}) \otimes H^1(C_n, \mathbb{Z}) \) or, using the intersection form, to a map \( \tau_n \in \text{End}(H_1(J(C_n), \mathbb{Z})) \). It is immediate that \( \tau_n \) is the map in homology induced by \( \lambda_n = f_n^* f_n^\ast : J(C_n) \to J(C_n) \), where, as usual, we denote by \( f_n^* \) and \( f_n^\ast \) the pullback map and the norm map induced by \( f_n \) on Jacobians. Since \( \lim_{n \to +\infty} T_n = T \), after identifying the homology of all the fibers of \( \pi \) to the homology of the central one, \( \tau_n \) becomes constant for large \( n \). In particular, setting \( t = c_1(T - dH_0) \), and denoting by \( \tau \) the corresponding endomorphism of \( H_1(J(C), \mathbb{Z}) \), we get that:

\[
\text{rank of } \tau = \text{rank of } \tau_n = 2g'.
\]

Moreover, \( \tau \) defines an endomorphism \( \lambda \) of \( J(C) \), which is the limit of the maps \( \lambda_n = f_n^* f_n^\ast \). Clearly the dimension of the image of \( \lambda \) is \( g' \). Composing \( \lambda_n \) with the difference map \( e_n \) we obtain maps \( \psi_n : C_n \times C_n \to J(C_n) \):

\[
\psi_n(x, y) = \lambda_n(\mathcal{O}(\mathcal{O}(x - y))).
\]

In the same way we may set \( \psi = \lambda e \), where \( e \) is the difference map \( C \times C \to J(C) \); clearly, \( \psi \) is the limit of the \( \psi_n \). Since by construction \( \psi_n(T_n) = 0_{J(C_n)} \), we get:

\[
\psi(T) = 0_{J(C)}.
\]

Now, composing \( \lambda \) with the Abel-Jacobi map, we obtain a morphism \( \eta : C \to J(C) \). To end the proof we study the map \( \tilde{\eta} : C \to \eta(C) \), and notice that:

i) the geometric genus of \( \eta(C) \) is bigger than one: in fact \( \eta(C) \) generates the image of \( \lambda \), so its geometric genus is at least \( g' \).

ii) \( \tilde{\eta} \) is not birational: in fact, for any \( (x, y) \in T, \eta(x) = \eta(y) \), and, as we have already observed, there is a component of \( T \) which is different from \( \Delta \).

In conclusion, \( C \) belongs to \( R \).

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REFERENCES


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