1. Holomorphic Functions of Several Variables

Today’s lecture will primarily be about complex manifolds and the extra structure they carry compared to a real manifold. To begin we will cover some preliminary notions from the theory of functions of several complex variables. Most of this material is from Voisin’s book [1].

Definition 1.1. A \( C^1 \) (continuously differentiable) function
\[ f : U \to \mathbb{C} \]
defined on an open subset \( U \subseteq \mathbb{C}^n \) is called holomorphic if the differential
\[ T_p f : T_p U \to T_{f(p)} \mathbb{C} = \mathbb{C} \]
is complex linear (as defined it is only real linear).

Now we will state some other characterizations of these functions.

Theorem 1.2. For a \( C^1 \) function \( f \) as above, the following are equivalent

- \( f \) is holomorphic.
- For all \( z_0 \in U \), there is a neighborhood, \( V \), of \( z_0 \) on which \( f \) is equal to a power series
  \[
  f(z + z_0) = \sum_{|I|=0}^{\infty} a_I z^I
  \]
  where \( I = (i_1, \ldots, i_n) \) is a multi-index, \( z^I = z_1^{i_1} \cdots z_n^{i_n} \), and \( |I| = i_1 + \ldots + i_n \). We require that there exist positive real numbers \( R_1, \ldots, R_n \) such that
  \[
  \sum |a_I| r^I < \infty
  \]
  for all \( 0 \leq r_k < R_k, \ k = 1, \ldots, n \).
- For any polydisk \( D = \{ z \in \mathbb{C}^n \mid |z_j - a_j| \leq \alpha_j \} \) contained in \( U \) and any \( z \) in the interior of \( D \), there is an equality
  \[
  f(z) = \left( \frac{1}{2\pi i} \right)^n \int_{|z_i - a_i| = \alpha_i} f(\zeta) \frac{d\zeta_1}{\zeta_1 - z_1} \wedge \cdots \wedge \frac{d\zeta_n}{\zeta_n - z_n}
  \]

Proof. We will briefly sketch the proof. The most important implication is that holomorphic functions satisfy Cauchy’s integral formula as in the last bullet point. This is simply proved the one-dimensional Cauchy formula and induction on the dimension, as the integral above can be considered an iterated integral. The other implications are standard analysis and can be found in Voisin’s book [1].

We will also need holomorphic functions whose target is a higher dimensional complex space, their definition is straightforward.

Definition 1.3. A holomorphic function \( f : \mathbb{C}^n \to \mathbb{C}^m \) is any \( C^1 \) function such that if \( f = (f_1, \ldots, f_m) \) then each \( f_j \) is holomorphic for \( j = 1, \ldots, m \).

Now we can define our main objects of study: complex manifolds.
Definition 1.4. A complex manifold of (complex) dimension $n$ is a topological space $M$ with an atlas $(U_\alpha, \varphi_\alpha)$ such that
\[ \varphi_\alpha : U_\alpha \to \mathbb{C}^n \]
is a homeomorphism from $U_\alpha$ to an open subset $V_\alpha$ of $\mathbb{C}_n$. Further, these charts are required to have the property that
\[ \varphi_\alpha \circ \varphi_\beta : (U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta) \]
is a biholomorphic map (holomorphic with holomorphic inverse) between the corresponding open sets of $\mathbb{C}^n$.

Definition 1.5. A complex manifold of (complex) dimension $n$ is a homeomorphism from $U_\alpha$ to an open subset $V_\alpha$ of $\mathbb{C}_n$. Further, these charts are required to have the property that
\[ \varphi_\alpha \circ \varphi_\beta : (U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta) \]
is a biholomorphic map (holomorphic with holomorphic inverse) between the corresponding open sets of $\mathbb{C}^n$.

2. Examples of complex manifolds

We will now show that any smooth affine complete intersection variety over $\mathbb{C}$ is a complex manifold, which gives a wealth of examples. We could remove the complete intersection condition, but the following proof applies most cleanly if we use that assumption.

Definition 2.1. Let $p_1, \ldots, p_n$ be polynomials in $m$ variables ($m \geq n$) with complex coefficients. Consider the map
\[ P : \mathbb{C}^m \to \mathbb{C}^n \]
such that $P = (p_1, \ldots, p_n)$. An complex affine variety is the set $V(p_1, \ldots, p_n) = P^{-1}(0)$. It is called a smooth complete intersection if at each $z \in V(p_1, \ldots, p_n)$ the matrix of partial derivatives
\[ T_z P = \begin{pmatrix} \frac{\partial p_1}{\partial z_1} & \cdots & \frac{\partial p_1}{\partial z_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial p_n}{\partial z_1} & \cdots & \frac{\partial p_n}{\partial z_m} \end{pmatrix} \]
has rank at least $n$.

To define holomorphic charts for such a variety, we need a holomorphic analogue of the implicit function theorem. This is Proposition 1.1.11 from the book *Complex Geometry* by Huybrechts [2].

Theorem 2.2. Let
\[ F : \mathbb{C}^m \to \mathbb{C}^n \]
be given by $n$ holomorphic functions $F = (f_1, \ldots, f_n)$ such that the truncated matrix of partial derivatives
\[ \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_1} & \cdots & \frac{\partial f_n}{\partial z_n} \end{pmatrix} \]
has rank $n$ at some point $z_0 \in \mathbb{C}^n$. Then we consider $\mathbb{C}^m = \mathbb{C}^{m-n} \times \mathbb{C}^n$ and then there are open sets $z_0 \in U$, $U' \subseteq \mathbb{C}^{m-n}$, $U'' \subseteq \mathbb{C}^n$, and a holomorphic function
\[ g : U' \to U'' \]
such that $U' \times U'' \subseteq U$, and $z \in U$ is in $F^{-1}(F(z_0))$ if and only if $z = (g(z_{n+1}, \ldots, z_m), z_{n+1}, \ldots, z_m)$, i.e. iff $z$ is in the graph of $g$.

This shows immediately that if $P$ satisfies the hypotheses of the implicit function theorem at a point $z_0 \in V(p_1, \ldots, p_n)$ then
\[ (z_{n+1}, \ldots, z_m) \mapsto (g(z_{n+1}, \ldots, z_m), z_{n+1}, \ldots, z_m) \]
gives a chart around $z_0$. By the hypotheses of smoothness, there are indices $i_1, \ldots, i_n$ such that
\[ \begin{pmatrix} \frac{\partial f_1}{\partial z_{i_1}} & \cdots & \frac{\partial f_1}{\partial z_{i_n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial z_{i_1}} & \cdots & \frac{\partial f_n}{\partial z_{i_n}} \end{pmatrix} \]
has rank $n$. By permuting the variables and then applying the implicit function theorem, we get a chart of the same form as above but with permuted indices. The transition functions are then given by either the identity or the holomorphic functions $g$ given by the theorem, and are thus holomorphic.

Another rich source of examples is complex tori. If $\Lambda \subseteq \mathbb{C}^n$ is a lattice, then $\mathbb{C}/\Lambda$ has a natural structure of complex manifold. The charts are just sufficiently small neighborhoods of points, and the transition functions are just translations. It takes a little work to show, but this gives examples that are not algebraic varieties.

3. Vector bundles on manifolds

Most of the constructs that are done in the theory of manifolds involve vector bundles, so we will cover the preliminary material on vector bundles now. We will assume the reader is familiar with smooth manifolds.

**Definition 3.1.** A smooth (real resp. complex) vector bundle of rank $r$ is a smooth map between real manifolds

$$\pi : E \rightarrow B$$

such that for any $b \in B$, $E_b = \pi^{-1}(\{b\})$ has the structure of a (real resp. complex) vector space of dimension $r$. We further require that for any $b \in B$ there is a neighborhood $b \in U$ such that there is a commutative diagram

$$
\begin{array}{ccc}
U \times F^r & \xrightarrow{\varphi} & \pi^{-1}(U) \\
\downarrow \pi_1 & & \downarrow \pi \\
U & \rightarrow & \pi^{-1}(U)
\end{array}
$$

where $\pi_1$ denotes the first projection, $F$ is $\mathbb{R}$ or $\mathbb{C}$, $\varphi$ is a $C^\infty$ map, and the map

$$\varphi_b : \{b\} \times \mathbb{R}^r \rightarrow E_b$$

is an $F$-linear isomorphism for each $b$.

**Example 3.2.** There are many examples. There are the trivial vector bundles

$$\mathbb{R}^r \times B \rightarrow B$$

as well as the tangent and cotangent bundles

$$TB \rightarrow B \quad \text{and} \quad \Omega^1(B) \rightarrow B.$$ We can also apply constructions from linear algebra to each fiber of a vector bundle to form new vector bundles. Namely if $E$ and $E'$ are two vector bundles over $B$, then

- The tensor product $E \otimes_{\mathbb{R}} E'$ is a bundle over $B$ whose fiber $(E \otimes_{\mathbb{R}} E')_b$ is equal to $E_b \otimes_{\mathbb{R}} E'_b$.
- The dual vector bundle $E^\vee$ exists and satisfies $E^\vee_b = (E_b)^\vee = \text{Hom}_{\mathbb{R}}(E_b, \mathbb{R})$
- The exterior powers $\Lambda^k E$ exist and satisfy $\left(\Lambda^k E\right)_b = \Lambda^k (E_b)$

In this framework we have $\Omega^1(B) = TB^\vee$ and there are also the bundles of $k$-forms $\Omega^k(B) = \Lambda^k \Omega^1(B)$.

**Definition 3.3.** A local section of a bundle $E$ over an open set $U \subseteq B$ is a smooth map $s : U \rightarrow E$ such that $\pi \circ s = id_U$. That is to each point $b \in U$, we assign a point in the fiber $E_b$. The collection of all such sections is denoted $E(U)$ and it has the structure of $\mathbb{R}$ or $\mathbb{C}$ vector space as well as the structure of a module over the ring $C^\infty(U)$ of smooth maps $U \rightarrow \mathbb{R}$.

**Definition 3.4.** A map between vector bundles is a commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\varphi} & E' \\
\downarrow \pi & & \downarrow \pi' \\
U & \rightarrow & U
\end{array}
$$

such that the induced maps on fibers

$$\varphi_b : E_b \rightarrow E'_b$$
are linear for each \( b \in B \). Some people require the maps \( \varphi_b \) to have constant rank as \( b \) varies, but I will not worry about this.

Sections of \( TB \) are called vector fields, and sections of \( \Omega^k(B) \) are called \( k \)-forms.

Because this class is about complex geometry, we will also need the notion of a holomorphic vector bundle over a complex manifold. First we will discuss the concept of transition functions. For \( \pi : E \to B \) a complex vector bundle of rank \( r \), consider a point \( b \in B \) with two trivializing neighborhoods \( U \) and \( U' \) as in the definition of a vector bundle. The composite map

\[
\varphi_{U,U'} = \varphi'^{-1} \circ \varphi : U \cap U' \times \mathbb{C}^r \to U \cap U' \times \mathbb{C}^r
\]

acts as the identity on the first factor and so it satisfies

\[
\varphi_{U,U'}(p,v) = (p, t_p(v))
\]

where for each \( p \in U \cap U' \), \( t_p \) is an element of \( GL_r(\mathbb{C}) \). As \( p \) varies this gives a map

\[
t : U \cap U' \to GL_r(\mathbb{C})
\]

called a transition map.

**Definition 3.5.** A complex vector bundle is called holomorphic if each transition function \( t \) is a holomorphic map. The total space \( E \) of a holomorphic bundle is a complex manifold such that \( \pi \) is a holomorphic map. Further a section \( s \in E(U) \) is called holomorphic if it is a holomorphic map between complex manifolds.

4. **Tangent bundles and Almost complex structures**

Let \( M \) be a complex manifold of complex dimension \( n \) and real dimension \( 2n \). For \( p \in M \), the tangent space \( T_pM \) has the structure of complex vector space, and therefore admits a linear automorphism given by multiplication by \( i \). These automorphisms glue together to give a vector bundle automorphism

\[
I : TM \to TM.
\]

On a local chart, there are holomorphic coordinates \( z_j = x_j + iy_j \), and as a real vector bundle, \( TM \) has the sections

\[
\frac{\partial}{\partial x_j} \quad \text{and} \quad \frac{\partial}{\partial y_j}
\]

for \( j = 1, \ldots, n \) which are a basis for \( T_pM \) at each point \( p \in M \). The action of \( I \) is given by

\[
I\left( \frac{\partial}{\partial x_j} \right) = \frac{\partial}{\partial y_j} \quad \text{and} \quad I\left( \frac{\partial}{\partial y_j} \right) = -\frac{\partial}{\partial x_j}
\]

Thus any local section is given by

\[
\sum_{j=1}^{n} f_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} g_j \frac{\partial}{\partial y_j}.
\]

This is enough for smooth manifolds, but if we are doing complex geometry, we want to use these vector fields to differentiate smooth functions \( M \to \mathbb{C} \). Thus the coefficients \( f_i \) and \( g_i \) should be allowed to be \( C^\infty \) functions \( M \to \mathbb{C} \). Vector fields with those coefficients are naturally sections of the complexified tangent bundle \( TM_\mathbb{C} = TM \otimes_\mathbb{R} \mathbb{C} \).

Furthermore, from the perspective of complex geometry, \( z_j \) and \( \bar{z}_j \) are more natural coordinates than \( x_j \) and \( y_j \). Thus we want to define vector fields \( \frac{\partial}{\partial z_j} \) and \( \frac{\partial}{\partial \bar{z}_j} \). These will be the dual basis of the local sections

\[
dz_j = dx_j + idy_j \quad \text{and} \quad \dbz_j = dx_j - idy_j
\]

for \( j = 1, \ldots, n \) of the complexified cotangent bundle \( \Omega^1(M)_\mathbb{C} = \Omega^1(M) \otimes_\mathbb{R} \mathbb{C} \).

It is interesting to note that at each point \( p \in M \), the functional

\[
d_pz_j : T_pM_\mathbb{C} \to \mathbb{C}
\]

is \( \mathbb{C} \)-linear because \( z_j \) is holomorphic, but

\[
d_p\bar{z}_j : T_pM_\mathbb{C} \to \mathbb{C}
\]
is in fact complex antilinear, i.e.
\[ dp(\bar{z}_j) \left( I \left( \frac{\partial}{\partial x_j} \right) \right) = (dp_x - id_p y_j) \left( \frac{\partial}{\partial y_j} \right) = -i = -ip \bar{z}_j \left( \frac{\partial}{\partial x_j} \right) \]
and similarly
\[ dp(\bar{z}_j) \left( I \left( \frac{\partial}{\partial y_j} \right) \right) = -id_p \bar{z}_j \left( \frac{\partial}{\partial y_j} \right). \]

The sections of \( \Omega^1(M) \) spanned by \( dz_j \) are therefore linear and constitute the "holomorphic part" \( \Omega^{1,0}(M) \). Similarly those spanned by the \( d\bar{z}_j \) are antilinear and constitute the "antiholomorphic part" \( \Omega^{0,1}(M) \). We get a direct sum decomposition
\[ \Omega^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M). \]

All of this dualizes to the tangent bundle. We have the identities
\[ \partial \partial z_j = \frac{1}{2} \left( \partial \partial x_j - i \partial \partial y_j \right) \quad \text{and} \quad \partial \partial z_j = \frac{1}{2} \left( \partial \partial x_j + i \partial \partial y_j \right) \]
and there is a splitting
\[ TM = T^{1,0}M \oplus T^{0,1}M \]
where the local sections \( T^{1,0}M(U_\alpha) \) are of the form
\[ \sum_{j=1}^{n} f_i \frac{\partial}{\partial z_j} \]
for smooth functions \( f_i : U_\alpha \rightarrow \mathbb{C} \) and some chart \( U_\alpha \). The analogous thing holds for \( T^{0,1} \).

It is useful to remark that the Cauchy-Riemann equations take on the simple form
\[ \frac{\partial f}{\partial \bar{z}} = 0 \]
For a complex valued \( C^\infty \) function of multiple complex variables, the equations
\[ \frac{\partial f}{\partial \bar{z}_j} = 0 \]
together imply that \( df \) lies in \( \Omega^{1,0} \) and therefore that \( f \) is holomorphic.

Note that there are automorphisms of real vector bundles
\[ TM \otimes_R \mathbb{C} \rightarrow TM \otimes_R \mathbb{C} \quad \text{and} \quad \Omega^1 \otimes_R \mathbb{C} \rightarrow \Omega^1 \otimes_R \mathbb{C} \]
given by conjugation on the \( \mathbb{C} \) part, and denoted with a bar. With this in mind,
\[ d\bar{z}_j = d\bar{z}_j, \quad \overline{\frac{\partial}{\partial z_j}} = \frac{\partial}{\partial \bar{z}_j} \]
and
\[ T^{1,0}M = T^{0,1}M, \quad \overline{\Omega^{1,0}(M)} = \Omega^{0,1}(M). \]

The discussion about \( dz \) being linear and \( d\bar{z} \) antilinear implies that \( T^{1,0}M \) and \( \Omega^{1,0}(M) \) are in fact the eigenspaces of \( I \) corresponding to the eigenvalue \( i \) and \( T^{0,1}M \) and \( \Omega^{0,1}(M) \) are the eigenspaces corresponding to the eigenvalue \( -i \).

Complex manifolds are very special even-dimensional real manifolds. For many constructions in complex geometry, the full structure of complex manifold is not necessary, and a weaker notion of almost complex structure can be used.

**Definition 4.1.** Let \( M \) be a smooth manifold and \( I : TM \rightarrow TM \) be an endomorphism such that \( I^2 = -id \).
Such a pair \((M,I)\) is called a manifold with an almost complex structure.
An almost complex structure gives the structure of complex vector space on each fiber $T_pM$, by the rule
$$(a + ib)v = av + bI(v).$$
This implies that $M$ is even dimensional, but it does not necessarily mean that $M$ is a complex manifold. Note that a complex manifold does have an almost complex structure where $I$ multiplies each fiber by $i$. An almost complex structure which actually comes from a complex manifold is called integrable.

If $(M, I)$ is an almost complex structure, then consider the complexified tangent bundle
$$TM \otimes_{\mathbb{R}} \mathbb{C}.$$ As before, $I$ has eigenspaces corresponding to $i$ and $-i$ and there is in fact a direct sum decomposition
$$TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M.$$ There is also the analogous splitting for the complexified cotangent bundle.

We want to classify the integrable almost complex structures, and to do this we first need the notion of bracket of vector fields.

**Definition 4.2.** Let $M$ be a real manifold and $\chi \in TM(U)$ be a vector field. This determines a map
$$\chi : C^\infty(U) \to C^\infty(U)$$
which in local coordinates is given by
$$\chi(f) = \sum_{j=1}^{n} \chi_j \frac{\partial f}{\partial x_j}.$$ When thought of as such maps, two vector fields $\chi$ and $\psi$ can be combined to make a new map
$$[\chi, \psi] = \chi \circ \psi - \psi \circ \chi.$$ This is the map associated to a unique vector field, also denoted by $[\chi, \psi]$.

**Theorem 4.3.** (Newlander-Nirenberg) An almost complex structure $(M, I)$ is integrable if and only if
$$[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M$$
that is, if and only if the bracket of two vector fields of $(0, 1)$ type is also of $(0, 1)$ type. The same is true for $(1, 0)$ type fields as well.

**Example 4.4.** Let $M$ be a 2 real dimensional compact oriented manifold. If $I$ is an almost complex structure on $M$ then the theorem shows that it is automatically integrable. Because $T^{0,1}M$ has complex rank 1, if $V \in TM^{0,1}(U)$ is a vector field that vanishes nowhere on $U$, then any other vector field $V'$ is equal to $fV$ for some unique complex valued function $f$. Then using a basic identity for brackets of vector fields (that is easily checked by hand)
$$[V, V'] = [V, fV] = f[V, V] + V(f)V = V(f)V \in T^{0,1}M(U).$$ With this proven, one shows that the existence of an almost complex structure on $M$ is equivalent to the existence of a Riemannian metric (which can always be constructed using partitions of unity), and therefore all compact orientable 2 real dimensional manifolds are complex manifolds.

5. Differential Forms

We now study further differential forms on complex manifolds. The decomposition
$$\Omega^1(M)_{\mathbb{C}} = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$$
also decomposes the exterior powers as
$$\bigwedge^k \Omega^1(M)_{\mathbb{C}} = \bigoplus_{p+q=k} \Omega^{p,q}(M)$$
where the bundles $\Omega^{p,q}$ are spanned in local coordinates by the sections
$$dz_f \wedge d\bar{z}_j.$$
where \( I = (i_1, \ldots, i_p) \) for \( 1 \leq i_1 < \ldots < i_p \leq k \), \( J = (j_1, \ldots, j_q) \) for \( 1 \leq j_1 < \ldots < j_q \leq k \), and
\[
dz_I = dz_{i_1} \wedge \ldots \wedge dz_{i_p} \quad \text{and} \quad d\bar{z}_J = d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q},
\]
As before we have \( \overline{\Omega^{p,q}(M)} = \Omega^{p',q}(M) \).

The real exterior derivative
\[
d : \Omega^k(M) \to \Omega^{k+1}(M)
\]
extends to a complex linear exterior derivative
\[
d : \Omega^k(M)_C \to \Omega^{k+1}(M)_C.
\]
It satisfies \( d \circ d = 0 \) as in the real case, and this implies that \( d(dz_j) = d(d\bar{z}_j) = 0 \), i.e. these forms are closed. In local coordinates, if \( f = u + iv \) is a complex valued \( C^\infty \) function, we compute the action of \( d \) on \((p, q)\)-forms as
\[
d(fdz_I \wedge d\bar{z}_J) = d(f) \wedge dz_I \wedge d\bar{z}_J = \sum_{j=1}^n \left( \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \right) \wedge dz_I \wedge d\bar{z}_J = \partial(fdz_I \wedge d\bar{z}_J) + \overline{\partial}(fdz_I \wedge d\bar{z}_J)
\]
Where
\[
\partial : \Omega^{p,q} \to \Omega^{p+1,q} \quad \text{and} \quad \overline{\partial} : \Omega^{p,q} \to \Omega^{p,q+1}
\]
are defined by
\[
\partial(fdz_I \wedge d\bar{z}_J) = \sum_{j=1}^n \left( \frac{\partial f}{\partial z_j} dz_j \wedge dz_I \wedge d\bar{z}_J \right) \quad \text{and} \quad \overline{\partial}(fdz_I \wedge d\bar{z}_J) = \sum_{j=1}^n \left( \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J \right).
\]
That is, \( d = \partial + \overline{\partial} \) and \( \partial \) is the \( \Omega^{p+1,q} \) part of \( d \) and \( \overline{\partial} \) is the \( \Omega^{p,q+1} \) part. The direct sum decomposition lets us translate identities for \( d \) into identities for \( \partial \) and \( \overline{\partial} \). In particular
\[
d \circ d = \partial \circ \partial + \overline{\partial} \circ \partial + \overline{\partial} \circ \overline{\partial} = 0
\]
The \((p + 2, q)\) part of this equation shows
\[
\partial \circ \partial = 0
\]
the \((p + 1, q + 1)\) part that
\[
\partial \circ \overline{\partial} + \overline{\partial} \circ \partial = 0
\]
and the \((p, q + 2)\) part that
\[
\overline{\partial} \circ \overline{\partial} = 0.
\]
We also have the Leibniz rule for \( \partial \) and \( \overline{\partial} \) by the same method. If \( \alpha \in \Omega^{p,q}(U) \) and \( \beta \in \Omega^{p',q'}(U) \) then
\[
d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{p+q} \alpha \wedge d\beta
\]
and the \((p + q, p' + q' + 1)\) part of this equation shows that
\[
\overline{\partial}(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^{p+q} \alpha \wedge \overline{\partial}\beta
\]
and the same for \( \partial \).

6. Local exactness

For smooth manifolds we have the Poincare lemma (for reference see Differential Forms in Algebraic Topology by Bott and Tu [3])

**Theorem 6.1.** Let \( \alpha \in \Omega^k(M) \) be a closed \( k \)-form, i.e. \( d\alpha = 0 \). Then on a neighborhood of each point, for example \( U \), there is some \( \beta \in \Omega^k(U) \) such that \( d\beta = \alpha \)

The point of this section is to show that the same thing happens for \( \overline{\partial} \).

**Proposition 6.2.** (Proposition 2.31 in [1]) Let \( \alpha \in \Omega^{p,q} \) satisfy that \( \overline{\partial}\alpha = 0 \). Then locally \( \alpha = \overline{\partial}\beta \).

**Proof.** For the proof we will need
Theorem 6.3. (Proposition 2.32 in [1]) If \( f(z_1, \ldots, z_n) \) is a \( C^1 \) function \( C^n \to \mathbb{C} \) that satisfies
\[
\frac{\partial f}{\partial \bar{z}_l} = 0
\]
for all \( l > q \) for some \( q < n \), then we say that \( f \) is holomorphic in the variables \( z_l \) for \( l > q \). If \( f \) is holomorphic in those variables then locally there exists a function \( g \), also holomorphic in \( z_l \) for \( l > q \), such that
\[
f = \frac{\partial g}{\partial \bar{z}_q}.
\]

We will only need the statement of that theorem, the proof is pure analysis.

Let
\[
\alpha = \sum_{I,J} \alpha_{I,J} dz_I \wedge d\bar{z}_J
\]
satisfy \( \overline{\partial} \alpha = 0 \). I claim that if
\[
\alpha_I = \sum_{J} \alpha_{I,J} d\bar{z}_J
\]
then \( \overline{\partial} \alpha_I = 0 \).

If this is true then because
\[
\alpha = \sum \alpha_I \wedge dz_I,
\]
we can prove the proposition for \( \alpha_I \) and it will imply it for \( \alpha \). More precisely if locally
\[
\alpha_I = \overline{\partial} \beta_I
\]
then
\[
\alpha = \sum_I dz_I \wedge \overline{\partial} \beta_I = \overline{\partial} (\sum_I (-1)^p dz_I \wedge \beta)
\]
by the Leibniz rule.

Now we will prove the claim.
\[
\overline{\partial} \alpha = \sum_I (-1)^p dz_i \wedge \overline{\partial} \alpha_I = 0
\]
As \( dz_I \wedge d\bar{z}_J \) forms a basis as \( I \) and \( J \) vary, it is necessary that each component of the above vanishes, that is
\[
dz_i \wedge \overline{\partial} \alpha_I = 0
\]
for each \( I. \) \( \alpha_I \) is a \((0, q)\) form, so \( \overline{\partial} \alpha_I \) is a \((0, q + 1)\) form, and therefore
\[
dz_i \wedge \overline{\partial} \alpha_I = 0 \quad \text{iff} \quad \overline{\partial} \alpha_I = 0
\]
This proves the claim.

Now we work with the \( \alpha_I \) instead of \( \alpha \) and assume that \( \alpha \) is a \((0, q)\) form. If
\[
\alpha = \sum_J \alpha_J d\bar{z}_J
\]
then we use induction on the integer \( k \) which is the largest integer appearing in the multi-index \( J \) corresponding to a non-zero \( \alpha_J \). Because \( 1 \leq j_1 < \ldots < j_q \), we must have that \( k \geq q \). For the base case we assume \( k = q \).

This means that
\[
\alpha = \alpha_J d\bar{z}_J
\]
for \( J = (1, \ldots, q) \). We know that \( \overline{\partial} \alpha_J = 0 \). Thus we can use the proposition above to say that locally, \( \alpha_J = \frac{\partial \beta_J}{\partial \bar{z}_q} \). Therefore
\[
\alpha = \overline{\partial}((-1)^{q-1} \beta_J d\bar{z}_1 \wedge \ldots \wedge d\bar{z}_{q-1})
\]
by the Leibniz rule.

Now assume we have proved it for some \( k > q \), and we will show it for \( k + 1 \).
We have that
\[
\alpha = \alpha_1 + \alpha_2 \wedge d\bar{z}_k
\]
where only the $d\bar{z}_j$ for $j < k$ appear in $\alpha_1$ and $\alpha_2$. Upon expanding $\partial \alpha$, we see that the only terms that contain a $d\bar{z}_k \wedge d\bar{z}_l$ for $l > k$ are coming from $\partial(\alpha_2 \wedge d\bar{z}_k)$ and thus if

$$\alpha_2 = \sum_J \alpha_{2,J} d\bar{z}_J$$

then

$$\frac{\partial \alpha_{2,J}}{\partial \bar{z}_l} = 0$$

for each $J$ and each $l > k$. Therefore each $\alpha_{2,J}$ is holomorphic in $z_l$ for $l > k$ and by the proposition above,

$$\alpha_{2,J} = \frac{\partial \beta_{2,J}}{\partial \bar{z}_k}$$

for functions $\beta_{2,J}$ which are holomorphic in the same variables.

Then

$$\partial(\sum_J (-1)^{q-1} \beta_{2,J} d\bar{z}_J) + \gamma = \alpha_2 \wedge d\bar{z}_k$$

where $\gamma$ only contains $d\bar{z}_l$ for $l < k$. Thus

$$\alpha = \alpha_1 + \gamma + \partial \beta.$$

Now we want to apply the induction hypothesis to $\alpha' = \alpha_1 + \gamma$. All we must do is show that it is $\partial$-closed. But we know

$$\partial \alpha = \partial \alpha' + \partial \partial \beta = \partial \alpha' = 0.$$

Thus locally, $\alpha' = \partial \beta'$ and

$$\alpha = \partial (\beta + \beta')$$

and we are done.

7. Dolbeault Complex of a holomorphic bundle

Let $\pi : E \to M$ be a holomorphic bundle, $M$ a complex manifold of complex dimension $n$. Denote by $A^{p,q}(E)$ the $C^\infty$ sections of $E \otimes_\mathbb{C} \Omega^{p,q}$, which we will call $E$-valued $p,q$ forms. If $s_1, \ldots, s_n$ are local holomorphic trivializing sections over an open set $U \subseteq M$ (so that the form a basis for the fiber at every point), then for $\alpha \in A^{p,q}(E)(U)$,

$$\alpha = \sum_i a_i s_i$$

for $a_i \in \Omega^{p,q}(U)$. We define

$$\partial_U \alpha = \sum_i (\partial a_i) s_i \in A^{p,q}(E)(U).$$

Let $t_{U,V} : U \cap V \to GL_r \mathbb{C}$ be the holomorphic transition function of $E$ for two trivializing open sets $U$ and $V$. Then we know that $\alpha_U \in A^{p,q}(E)(U)$ and $\alpha_V \in A^{p,q}(E)(V)$ agree on $U \cap V$ if and only if

$$a_{j,V} = \sum_k M^{j,k}_{U,V} a_{k,U}.$$

Now we want to see that if

$$\alpha_U|_{U \cap V} = \alpha_V|_{U \cap V}$$

then

$$\partial_U \alpha_U|_{U \cap V} = \partial_V \alpha_V|_{U \cap V}$$

We do this by using the Leibniz formula and the fact that $M^{j,k}_{U,V}$ are holomorphic, so that

$$\partial a_{j,V} = \sum_k M^{j,k}_{U,V} \partial a_{k,U}$$

as desired.

By glueing these operators together, we get an operator

$$\partial_E : A^{p,q}(E) \to A^{p,q+1}(E)$$
The holomorphic sections of $E$ are exactly those with homorphic coefficients in a local chart, which by the local description of $\overline{\partial} E$ are exactly the functions in the kernel of $\overline{\partial} E$.

We have the analogue of the Leibniz formula, along with $\overline{\partial}^2 E = 0$, and local exactness all by using the local description of $\overline{\partial} E$. This gives a complex of vector bundles

$$0 \to E = A^{0,0}(E) \to A^{0,1}(E) \to ...$$

which is called the Dolbeault complex of $E$. In the language of sheaves, the local exactness means this is an exact sequence of sheaves.

References