KÄHLER METRICS

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Introduction. I will closely follow the content and notations of chapter three in the book [1, Hodge Theory and Complex Algebraic Geometry, I]. This is a lecture note for two consecutive talks.

In the first part, we will describe additional structures on complex manifolds. In particular, we will define hermitian metric and study its relation with the Kähler forms, which is a 2-form of certain type. This leads to defining certain connections (Levi-Cevita and Chern connection) on the manifold and we will provide a characterization of Kähler Manifold in terms of these connections.

In the second part we will be providing examples and ways to construct Kähler manifolds. We will show that Riemann surfaces, Complex Tori, Projective spaces are Kähler. Moreover we shall also show that Projective bundles and Blow ups over Kähler manifolds are Kähler.

1. Definition and basic properties

1.1. Hermitian forms. Let $V$ be a complex vector space, which will be often considered as a real vector space with the natural endomorphism $I$. Let $W_\mathbb{R} = \text{Hom}(V, \mathbb{R})$ and $W_\mathbb{C} = \text{Hom}(V, \mathbb{C})$ be the space of 1-forms. We have seen in the previous lecture that $W_\mathbb{C}$ can be decomposed as $W^{1,0} \oplus W^{0,1}$ in the $\mathbb{C}$-linear and $\mathbb{C}$-anti-linear forms. Let $W^{1,1} = W^{1,0} \otimes W^{0,1} \subset \bigwedge^2 W_\mathbb{C}$ and $W^{1,1}_\mathbb{R} = W^{1,1} \cap W_\mathbb{R}$.

For motivation purpose, one should think of the $V$ as a tangent space and $W$ as its dual. For example, let $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ be real basis of $V$ (1 complex dimensional), then endomorphism $I$ is given by $I(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}$ and $I(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial x}$. So $W_\mathbb{C}$ has basis $\{dx, dy, idx, idy\}$ which splits as a direct sum of $W^{1,0} = \mathbb{C}dz$ and $W^{0,1} = \mathbb{C}d\bar{z}$, where $dz = dx + idy$ and $d\bar{z} = dx - idy$. This gives us $W^{1,1} = \mathbb{C}dz \wedge d\bar{z} = \mathbb{C}dx \wedge dy$ and $W^{1,1}_\mathbb{R} = \mathbb{R}dx \wedge dy$ is a real one dimension vector space.

Definition 1.1. A hermitian form $h$ is a function $h : V \times V \to \mathbb{C}$ which is complex linear on left, complex anti-linear on right argument and satisfies $h(u, v) = \overline{h(v, u)}$. 

Lemma 1.2. There is a natural identification between the hermitian forms, $h$, on $V \times V$ and elements $w \in W_{\mathbb{R}}^{1,1}$ given by

\[(1) \quad h \rightarrow w = -\mathfrak{I}h\]

where $\mathfrak{I}$ means the imaginary part, that $h(u,v) = g(u,v) - iw(u,v)$ where $g$ and $w$ are real valued.

Proof. Since $h(u,v) = \overline{h(v,u)}$, we get $w(u,v) = -w(v,u)$, thus we have $w \in W_{\mathbb{R}}$. Let $V^{1,0}$ and $V^{0,1}$ be sub-spaces of $V_{\mathbb{C}} = V \otimes \mathbb{C}$ as defined in the previous lecture. We also use that fact that $V^{1,0}$ and $V^{0,1}$ is generated by elements of the form $v - iIv$ and $v + iIv$ respectively. Observe that $w \in W^{1,0} \otimes W^{0,1}$ iff $w(u,v) = 0$ for both $u,v$ in $V^{1,0}$ or in $V^{0,1}$. So we need to show that $w(u - iIu, v - iIv) = 0$, which is equivalent to showing that $w(u,v) = w(Iu, Iv)$ and $w(Iu, v) + w(u, Iv) = 0$, which follows from hermitian property of $h$. Similarly $w(u,v) = 0$ for $u,v \in V^{0,1}$.

Given an element $w$ in $W_{\mathbb{R}}^{1,1}$, we can define $g(u,v) = w(u,Iv)$, and $h(u,v) = g(u,v) - iw(u,v)$. Thus $h$ is $\mathbb{C}$ anti-linear on the right since $h(u, Iv) = g(u, Iv) - iw(u, Iv) = -w(u,v) + ig(u,v) = -ih(u,v)$ and similar calculation show $\mathbb{C}$ linearity on left. Since $w$ is alternating, the following calculation finishes the proof by noting $g$ is symmetric

$$h(v,u) = g(v,u) - iw(v,u) = g(v,u) + iw(u,v) = \overline{h(v,u)}$$

Definition 1.3. We say that a real alternating form $w$ of type $(1,1)$ is positive is the corresponding Hermitian form $h$ is positive definite.

Remark 1.4. Note that in the proof of lemma 1.2, we could have naturally identified a symmetric bilinear form $g$ to any Hermitian metric $h$. To have a bijection, we need to impose the condition $g(Iu, Iv) = g(u,v)$.

1.2. Hermitian and Kähler metric. Let $M$ be a manifold with almost complex structure $I$. A Hermitian metric $h$ on $M$ is a collection of of Hermitian metric $h_x$ on each fiber of the tangent bundle $T_{h^1, x}$ (complex structure given by $I_x$). We say $h$ is continuous/differentiable if in a local coordinate $\{x_1, x_2, \ldots, x_n\}$, the functions $x \rightarrow h_x(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_j})$ is continuous/differentiable.

Using lemma 1.2, we obtain a 2-form $w \in \Omega^{1,1}M \cap \Omega^2_{\mathbb{R}} M$, we call this Kähler form of the metric $h$.

Definition 1.5. We say a hermitian metric $h$ is a Kähler metric if $I$ is positive definite, integrable and the corresponding Kähler form $w$ is closed.
1.3. **Volume form.** Given a complex manifold $M$ (or a manifold having almost complex structure), we have a canonical way of defining an orientation on $M$. For the tangent space $T_{M,m}$, let $u_1, u_2, \ldots, u_n$ be a complex basis, then $(u_1, I(u_1), u_2, \ldots, I(u_n))$ is the oriented basis for $T_{M,m}$ over $\mathbb{R}$. The orientation does not depend on the choice of the basis.

Let $h$ be a Hermitian metric on $M$, it automatically endow $M$ with a Riemannian metric given by $g$ as defined in lemma 1.2. An oriented Riemannian manifold has a canonical volume form which is a non-degenerate section of $\Omega^2_{M}$, such that it yields a positive number for every oriented basis of $T_{M,m}$ at any point $m \in M$.

**Lemma 1.6.** The volume form associated to the Hermitian metric $h$ is $\frac{w^n}{n!}$.

**Proof.** Let $e_1, e_2, \ldots, e_n$ be a basis for $T_{M,m}$ over $\mathbb{C}$ such that $h(e_i, e_j) = \delta_{i,j}$. Then $e_1, Ie_1, e_2, \ldots, Ie_n$ is an orthonormal basis for $g_m$ with positive orientation. Let $dx_1, dy_1, \ldots, dy_n$ be its dual basis, and $dz_j = dx_j + idy_j$, then using the facts $w(e_i, e_j) = 0 = w(Ie_i, Ie_j)$ and $w(e_i, Ie_j) = -id\delta_{i,j} = -e(Ie_i, e_j)$, we get that $w = \frac{i}{2} \sum_{k=1}^{n} dz_k \wedge d\bar{z}_k$.

So we have

$$\frac{w^n}{n!}(e_1, Ie_1, e_2, \ldots, Ie_n) = \frac{i^n}{2^n} dz_1 \wedge \cdots \wedge d\bar{z}_n(e_1, Ie_1, e_2, \ldots, Ie_n).$$

Note that $dz_k \wedge d\bar{z}_k = -2idx_k \wedge dy_k$, hence the above expression is 1. Thus $\frac{w^n}{n!}$ is the volume form. \hfill \Box

**Corollary 1.7.** If $M$ is a compact Kähler manifold, then for every integer $k \in \{1, 2, \ldots, n\}$, the closed form $w^k$ is not exact (i.e $w \neq d\gamma$).

**Proof.** Suppose $w^k = d\gamma$, then $w^n = d(w^{n-k} \wedge \gamma)$. So by Stokes theorem,

$$\int_M w^n = \int_{\partial M} d(w^{n-k} \wedge \gamma) = 0$$

which is contradictory since it is integral of $n!$ times the volume form, which is positive. \hfill \Box

**Remark 1.8.** The previous corollary implies that any compact Kähler manifold has a non-trivial De Rham cohomologies of even degree. This fact can be used to show existence of non-Kähler manifolds.

2. **Characterizations of Kähler metrics**

2.1. **Background on Connections.** Let $E$ be a vector bundle over a differential manifold $M$. A connection on $E$ is a map $\nabla : C^\infty(E) \to$
$A^1(E)$, where $A^k(E)$ is the $C^\infty$ sections of the bundle $\Omega^k_{M,\mathbb{R}} \otimes E$, such that it satisfy Leibniz’ rule

$$\nabla f\sigma = df \otimes \sigma + f\nabla \sigma.$$ 

If $\psi$ is a vector fields, we define $\nabla_\psi \sigma = (\nabla \sigma)(\psi) \in C^\infty(E)$. Let $U$ be a subset of $M$ where $E$ trivializes, that is $E_U = U \times \mathbb{R}^k$. Then a section $\sigma$ of $E_U$ can be written as $\sum_{j=1}^k f_j e_j$, where $f_j \in C^\infty(U)$. Thus

$$\nabla \sigma = \sum_{j=1}^k df_j \otimes e_j + (f_1, f_2 \ldots f_k) \cdot B$$

where $B$ is the $k \times k$ matrix $\{b_{i,j}\}$, such that $\nabla e_j = \sum_{i=1}^k b_{i,j} \otimes e_j$. Note that $b_{i,j}$ is a section of $\Omega^1_{M,\mathbb{R}}$.

Let $(M, g)$ be a Riemannian manifold, then we say a connection $\nabla$ on $T_{M,\mathbb{R}}$ is compatible with the metric $g$, if it satisfies

$$d(g(\chi, \psi)) = g(\nabla \chi, \psi) + g(\chi, \nabla \psi)$$

where the right hand side is a 1-form, by assuming linearity of $g$ (i.e $g(u, \alpha \otimes v) = \alpha g(u, v)$). The following proposition is a well known fact from differential geometry

**Proposition 2.1.** If $(M, g)$ is a Riemannian manifold, there is a unique connection (called Levi-Cevita connection)

$$\nabla : T_{M,\mathbb{R}} \to A^1(T_{M,\mathbb{R}})$$

such that it is compatible with the metric $g$ and symmetric (that is $\nabla \chi \psi - \nabla \psi \chi = [\chi, \psi]$).

We will now consider holomorphic bundle $E$ over a complex manifold. Suppose that $E$ is equipped with Hermitian metric $h$ (defined in natural fashion). In the previous lecture we defined the operator

$$\bar{\partial}_E : C^\infty(E) \to A^{0,1}(E)$$

where $A^{0,q}$ is the $C^\infty$ section of $\Omega^{0,q}_M \otimes E$. Moreover we had the Leibniz’ rule given by

$$\bar{\partial}_E(\alpha \wedge \beta) = \bar{\partial}_E \alpha \wedge \beta + (-1)^q \alpha \wedge \bar{\partial}_E \beta.$$ 

Let $\nabla$ be a complex connection on $E$, then we can define

$$\nabla^{0,1} : C^\infty(E) \to A^{0,1}(E)$$

obtained by composing $\nabla$ with the projection $A^1(E) \to A^{0,1}_E$. we say that a connection $\nabla$ is compatible with $h$ if we have

$$d(h(\sigma, \tau)) = h(\nabla \sigma, \tau) + h(\sigma, \nabla \tau),$$

(3)
where $\sigma$ and $\tau$ are sections of $E$. Note that we define $h(u, \alpha \otimes v) = \bar{\partial}h(u, v)$.

**Proposition 2.2.** There is a unique connection (called Chern connection of $(E, h)$)

$$\nabla : C^\infty(E) \to A^1(E)$$

such that it is compatible with the metric $h$ and we have

$$\nabla^{0,1} = \bar{\partial}_E$$

**Proof.** Looking at $(1,0)$ part of equation 3, we get

$$\partial(h(\sigma, \tau)) = h(\nabla^{1,0}\sigma, \tau) + h(\sigma, \bar{\partial}_E\tau)$$

since $\nabla^{0,1} = \bar{\partial}_E$. Let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be holomorphic local basis of $E$, then $\bar{\partial}_E(\sigma_i) = 0$, thus $\nabla$ is uniquely determined by

$$h(\nabla^{1,0}\sigma_i, \sigma_j) = \partial(h(\sigma_i, \sigma_j)).$$

\[ \square \]

### 2.2. Kähler metric and connection

Let $X$ be a complex manifold and $h$ be a Hermitian metric on the complex tangent bundle $T_X$. From the proof of lemma 1.2, we have see that if $h = g - iw$, then $g$ is a Riemannian metric on $X$. So we have Levi-Cevita connection on $T_{X,\mathbb{R}}$ corresponding to $(X, g)$ and Chern connection on $T_X$ corresponding to $(X, h)$.

The next theorem characterizes Kähler metric in terms of ‘compatibility’ of the two connections.

**Theorem 2.3.** The following properties are equivalent:

(i) The metric $h$ is Kähler.

(ii) The complex structure endomorphism $I$ is flat for the Levi-Cevita connection. This means that it satisfies

$$\nabla(I\chi) = I\nabla\chi$$

(iii) The Chern connection and the Levi-Cevita connection coincide on $T_X$, identified with $T_{X,\mathbb{R}}$ via the map $\Re$.

**Proof.** (iii) $\implies$ (ii) : It follows because the Chern connection is $\mathbb{C}$-linear and $\Re$ identifies multiplication by $i$ with the endomorphism $I$.

(ii) $\implies$ (i) : Note that we have $g(u, v) = w(u, Iv)$. Using this and the fact that $\nabla$ and $I$ commutes we get

$$d(w(\chi, \psi)) = w(\nabla\chi, \psi) + w(\chi, \nabla\psi).$$
thus for any vector field $\phi$, we have
\[
\phi(w(\chi, \psi)) = w(\nabla_\phi \chi, \psi) + w(\chi, \nabla_\phi \psi).
\]

We use the formula describing $dw$ given as
\[
dw(\phi, \chi, \psi) = \phi(w(\chi, \psi)) - \chi(w(\phi, \psi)) + \psi(w(\phi, \psi)) - w([\phi, \chi], \psi) + w(\phi, [\chi, \psi]) + w([\phi, \psi], \chi).
\]
Expanding the first three terms using the previous equation, and symmetric property of Levi-Cevita connection, we find out that $dw = 0$.

$(i) \implies (iii)$: Firstly note that the Chern connection and Levi-Cevita connection coincide for $C^n$ for the Kähler metric associated to identity hermitian metric (i.e $h = \sum_{i=1}^{n} dz_i d\bar{z}_i$). Moreover note that in local coordinate, using equation (6), Chern connection depends only on the matrices of the metric to the first order. This is true for the Levi-Cevita connection as well. So if we show that for any point, there is a neighborhood and local coordinates $z_i$'s such that the matrix $h_{ij} = I_n + O(\sum_{i=1}^{n} z_i^2)$ up to order 2 terms, then the two connection coincide.

This is the content for the next proposition. □

**Proposition 2.4.** Let $h$ be a Kähler metric on an $n$ dimensional complex manifold $X$, then, in a neighborhood of each point $x$, there exist holomorphic coordinates $z_1, \ldots, z_n$, centered at $x$, such that the matrix $h_{ij}$ equals $I_n + O(\sum_{i=1}^{n} z_i^2)$

**Proof.** By basic linear algebra, we may take holomorphic coordinates $z_1, \ldots, z_n$ center at $x$, such that $h_x = I_n$ (as matrix) in the basis $\frac{\partial}{\partial z_i}$. So
\[
h = \sum_{i=1}^{n} dz_i d\bar{z}_i + \sum_{i,j} \epsilon_{ij} d z_i d\bar{z}_j + O(|z|^2)
\]
where $\epsilon_{ij}$ are linear forms in the $z_i, \bar{z}_i$. We can write $\epsilon_{ij} = \epsilon_{ij}^{\text{hol}} + \epsilon_{ij}^{\text{antihol}}$. Since $h$ is Hermitian, we have $\epsilon_{ij}^{\text{antihol}} = \epsilon_{ij}^{\text{hol}}$. So we can write
\[
w = \frac{i}{2} \left( \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i + \sum_{i,j} \epsilon_{ij}^{\text{hol}} dz_i \wedge d\bar{z}_j + \sum_{i,j} \epsilon_{ij}^{\text{antihol}} dz_i \wedge d\bar{z}_j + O(|z|^2) \right)
\]
and $dw = 0$ implies $\sum_{i,j} \epsilon_{ij}^{\text{hol}} dz_i \wedge d\bar{z}_j$ is $\partial$ closed.

We thus have $\frac{\partial}{\partial z_k} \epsilon_{ij} = \frac{\partial}{\partial z_i} \epsilon_{kj}$, and using basic facts about solving PDE, there exist $\phi_j$, such that $\epsilon_{ij} = \frac{\partial}{\partial z_i} \phi_j$. We may assume that $\phi_j$'s are
zero at $x$. Now we can define new coordinates given as $z'_i = z_i + \phi_i(z)$. Then a simple calculation show that

$$w = \frac{i}{2} \left( \sum_{i=1}^{n} dz'_i \wedge d\bar{z}'_i + + O(|z'|^2) \right)$$

which gives us required result about the corresponding Hermitian form. \hfill \Box

3. Examples of Kähler Manifolds

We note that Riemann surfaces and complex tori are Kähler manifold for obvious reasons. First one is because of dimension and the second follows for the fact that it is a quotient of $\mathbb{C}^n$ by a discreet subgroup, and part (ii) of theorem 2.3. Also note that a complex sub manifold naturally inherits a Kähler metric.

In this section, we will show that complex projective space is Kähler, hence all projective varieties are Kähler. Moreover we will show that Projective bundle over a Kähler manifold and blow up of sub manifolds are Kähler.

3.1. Chern Form. Let $X$ be a complex manifold and $L$ be a holomorphic line bundle with a Hermitian form $h$. Observe that over any fiber $L_x$, $h_x$ is defined by value $h_x(u, u)$ (by abuse of notation call it $h(u)$) for any (basis) element in one dimensional $\mathbb{C}$-vector space $L_x$. So for any trivializing neighborhood $U_i$, we can define function $h_i(x) = h_x(\sigma_i)$, where $\sigma_i$ is a chosen section of the restriction of $L$.

We know that $\sigma_i$ and $\sigma_j$ obtained from $U_i$ and $U_j$ are related as $\sigma_i = g_{ij} \sigma_j$, where $g_{ij}$ are invertible holomorphic functions on the intersection $U_{ij}$. Thus $h_i = |g_{ij}|^2 h_j$, by hermitian property of $h$.

Let $w$ be a 2-form of type $(1,1)$ defined as

$$w_j = \frac{1}{2\pi j} \partial \bar{\partial} \log h_i.$$ 

Note that it is well defined since $\partial \bar{\partial} \log |g_{ij}|^2 = 0$. Form $w$ is closed and real of type $(1,1)$.

Remark 3.1. When line bundle $L$ satisfy some positivity condition, we can construct Hermitian metric $h$ on $L$, such that $w$ associated to $L$ is positive.

3.2. Fubini-Study Metric. Let $\mathbb{P}^n$ be the complex projective space. Over $\mathbb{P}^n$, we have a natural line bundle $S$ which is a subbundle of $\mathbb{P}^n \times \mathbb{C}^{n+1}$ containing elements of the form $(\Delta, z)$, where $\Delta$ is a 1-dim linear subspace of $\mathbb{C}^{n+1}$ and $z \in \Delta$. 


Let $h$ be the standard metric on $\mathbb{C}^n$, therefore $S$ inherits an Hermitian metric and so does $O_{\mathbb{P}^n}(1) = S^\vee$ (call the metric $h^\ast$). Let the Chern form associated to $O_{\mathbb{P}^n}(1)$ be $w$, which is a $(1,1)$ real form.

For any trivialisation of $O_{\mathbb{P}^n}(1)$, let $\sigma_j$ be holomorphic section over open sets $U_j$. Then

$$w_j = \frac{1}{2\pi i} \partial \bar{\partial} \log h^\ast(\sigma_j^\ast)$$

where $\sigma_j^\ast$ is the dual to $\sigma_j$ and $\sigma_j(z_1, \ldots, z_n) = (z_1, \ldots, 1, z_j, \ldots, z_n)$. We have $h_j(\sigma_j) = 1/h^\ast_j(\sigma_j^\ast)$, and $h(\sigma_j) = 1 + \sum_{i=1}^n |z_i|^2$. Thus expanding the formula for $w_i$, we get

$$w_j = \frac{i}{2\pi} \frac{(1 + \sum_i |z_i|^2) \sum_i dz_i \wedge d\bar{z}_i - (\sum_i \bar{z}_i dz_i) \wedge (\sum_i z_i d\bar{z}_i)}{(1 + \sum_i |z_i|^2)^2}.$$ 

We observe that at $z = 0$, it is standard Hermitian metric hence positive. Since these open sets in $\mathbb{P}^n$ centered at any point, we have proved the following

**Lemma 3.2.** The form $w$ on $\mathbb{P}^n$ is positive, real, closed and of type $(1,1)$, hence $\mathbb{P}^n$ Kähler.

**Definition 3.3.** Let $E$ be a holomorphic vector bundle of rank $r+1$ on a complex manifold $X$. We define $\mathbb{P}(E)$ as topological space obtained by taking quotient of $E - \{\text{zero section}\}$ by the action of $\mathbb{C}^\ast$.

There is a natural complex structure on $\mathbb{P}(E)$, and a projection map $\pi : \mathbb{P}(E) \to X$ such that $\pi^{-1}(x) = \mathbb{P}(E_x) \equiv \mathbb{P}^r$. We can define the line bundle $O_{\mathbb{P}(E)}(1)$ as in the case of projective space. Suppose $h$ is a metric on $E$, then it induced a metric on $O_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$. Let $w_E$ be its Chern form on $\mathbb{P}(E)$.

In addition, let us assume that $X$ is Kähler. Then there is another $(1,1)$ form on $\mathbb{P}(E)$ which is obtained by pulling back $w_X$ to $\mathbb{P}(E)$ (i.e $\pi^*(w_X)$).

**Proposition 3.4.** If $X$ is compact Kähler, and $E$ is a holomorphic bundle (let $h$ be Hermitian metric on $E$), then $\mathbb{P}(E)$ is Kähler.

**Proof.** Note that $\pi^*(w_X)$ is a semi positive form on $\mathbb{P}(E)$ and it vanishes exactly on the tangents to the fibers $\pi^{-1}x$. While we can show that $w_E$ is positive on the fibers (since it matches the form associated to Fubini-Study metric). Thus because of compactness we can choose $\lambda >> 0$ such that the form $w = w_E + \lambda \pi^*w_X$ is positive. □

**Remark 3.5.** In the similar fashion as above, we can show that for any complex submanifold $Y \subset X$, The blowup $\tilde{X}_Y$ is Kähler if $X$ is Kähler. Note that we have $\tau : \tilde{X}_Y \to X$, such that $\tau$ is isomorphism.
away from \( \tau^{-1}(Y) \), and \( \tau^{-1}(Y) = \mathbb{P}(N_{Y/X}) \). So we have a positive form on fibers, which we glue using the line bundle \( O(-D) \), where \( D \) is the exceptional divisor. This gives us a form \( w_D \), and obviously we have a form \( \tau^* w_X \). This gives us a positive form \( C \tau^* w_X + w_D \) where we take \( C >> 0 \) suitably chosen. For the detailed description of the \( w_D \) please refer to [1].

**References**