Differentiable manifolds:

Idea: generalize regular surfaces in $\mathbb{R}^3$.

Recall: A regular surface in $\mathbb{R}^3$ is a subset $M \subset \mathbb{R}^3$ s.t. $\forall x \in M$ there exists a neighborhood $V$ of $x$ in $\mathbb{R}^3$ and a mapping $\varphi: U \rightarrow V \cap M$ from an open subset $U$ of $\mathbb{R}^2$ s.t.

(a) $\varphi$ is a differentiable homeomorphism

(b) the differential $(d\varphi)_x: T_x U \rightarrow T_{\varphi(x)} \mathbb{R}^3$

is injective $\forall x \in U$.

Def: $\varphi: U \rightarrow V \cap M$ as above is called a coordinate chart on $M$ near $x$.

Given a regular surface $M \subset \mathbb{R}^3$ and two coordinate charts $\varphi: U \rightarrow M, \psi: V \rightarrow M$ s.t. $W = \varphi(U) \cap \psi(V) \neq \emptyset$ the map $\psi^{-1} \circ \varphi: \varphi^{-1}(W) \rightarrow \mathbb{R}^2$ is differentiable
Definition: A differentiable manifold of dimension $n$ is a set $M$ together with the data of a set of open subsets $\{U_\alpha\}$ of $\mathbb{R}^n$ and injective maps $\varphi_\alpha: U_\alpha \to M$ s.t.:

1) $\bigcup \varphi_\alpha(U_\alpha) = M$
2) $\forall \alpha, \beta$ s.t. $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) \neq \emptyset$

Warp.

$\varphi_\beta^{-1} \circ \varphi_\alpha: \varphi_\alpha^{-1}(\text{Warp}) \to \mathbb{R}^n$

is differentiable.

Remark: The differentiable structure induces a topology on $M$: a subset $W$ of $M$ is open $\iff$ $\forall \alpha$, $\forall x \in \varphi_\alpha^{-1}(W \cap \varphi(U_\alpha))$ is open.
In this topology \( P(V_{\mathcal{A}}) \) is open \( \forall \alpha \) & \( y_{\mathcal{A}} \) is continuous \( \forall y_{\mathcal{A}} \).

**Example:** Real projective space \( P^n(R) \):

**Def.:** \( P^n(R) := \frac{R^{n+1} \setminus \{0\}}{R^*} \).

i.e., \( P^n(R) \) is the set of equivalence classes for the equivalence relation:

\[
(x_1, \ldots, x_{n+1}) \sim (y_1, \ldots, y_{n+1})
\]

\[
\exists \lambda \in R^* \quad \exists \alpha \in R^* \quad \forall i \quad y_i = \lambda x_i + \alpha.
\]

Equivalently, \( P^n(R) \) is the set of lines through the origin \( v \in R^{n+1} \).

\( P^n(R) \) is a differentiable manifold:

Define \( V_i \subseteq P^n \) as the set of equivalence classes \( \{x_1, \ldots, x_{n+1}\} \) s.t. \( x_i \neq 0 \).
note that \( \bigcup_{i=1}^{n+1} V_i = \mathbb{P}^n \)

also \([x_1, \ldots, x_{n+1}] = \left[ \frac{x_1}{x_i}, \ldots, \frac{x_{n+1}}{x_i} \right]\)

\(\psi_i : \mathbb{R}^n \rightarrow \mathbb{P}^n\)

\((y_1, \ldots, y_n) \mapsto [y_1, \ldots, 1, \ldots, y_n]\)

\(\psi_i\) is injective.

to have a differentiable structure, we need \(\forall i, j\)

\(\psi_j \circ \psi_i : \psi_i^{-1}(V_i \cap V_j) \rightarrow \mathbb{R}^n\)

to be differentiable.

\(V_i \cap V_j = \{ (x_1, \ldots, x_n) \text{ homogeneous coordinates s.t. } x_i \neq 0, x_j \neq 0 \}\)

\(\psi_i (y_1, \ldots, y_n) = [y_1, \ldots, 1, \ldots, y_n]\)

\(y_j \neq 0\)

(WLOG: \(i \neq j\))

\(\psi_j^{-1}(V_i \cap V_j) = \{ (y_1, \ldots, y_n) : y_j \neq 0 \}\)

\((\phi_j^{-1} \circ \phi_i) (y_1, \ldots, y_n) = \psi_i^{-1} [y_1, \ldots, 2, \ldots, y_n]_i\text{-th.}\)
\[
\varphi_j(\mathbf{z}_1, \ldots, \mathbf{z}_n) = [z_1, \ldots, 1, \ldots, z_n] \\
\text{\textup{j-th}} \\
(\mathbf{z}_1, \ldots, \mathbf{z}_n) = \varphi_j^{-1}[z_1, \ldots, 1, \ldots, z_n] \\
[z_1, \ldots, 1, \ldots, z_n] = \left[\frac{z_1}{z_1}, \ldots, \frac{1}{z_i}, \ldots, z_n\right] \\
\text{So } \varphi_j^{-1}(\mathbf{y}_1, \ldots, \mathbf{y}_n) = \varphi_j^{-1}[\mathbf{y}_1, \ldots, 1, \ldots, \mathbf{y}_n] \\
= \left(\frac{\mathbf{y}_1}{\mathbf{y}_i}, \ldots, \frac{\mathbf{y}_n}{\mathbf{y}_i}\right) \\
= \left(\frac{y_1}{y_i}, \ldots, \frac{y_i-1}{y_i}, \frac{y_i+1}{y_i}, \ldots, \frac{y_n}{y_i}\right)
\]

is differentiable where \(y_i \neq 0\)

**Differentiable maps:**

**Def:** Given two manifolds \(M_1, M_2\), a map \(\varphi : M_1 \rightarrow M_2\) is differentiable if \(\forall \mathbf{z} \in M_1, \ \exists \ V \subset \mathbb{R}^n\)

\(g : V \rightarrow M_2\) a coordinate chart s.t. \(g(V) = \varphi(\mathbf{z})\)

\(\exists \ f : U \rightarrow M_1, \ V \subset \mathbb{R}^n\) coordinate chart of \(M_1\) s.t. \(f \circ \varphi^{-1}(V)\)
Define \( f : U \to M \), \( U \subseteq \mathbb{R}^n \)
and \( \varphi (f(U)) \subseteq g(V) \)
and \((g' \circ f) : U \to V \subseteq \mathbb{R}^n\)
is differentiable.

Note: With this definition, any coordinate chart \( \varphi : U \to M \)
is differentiable.

**Def:** A differentiable curve is a differentiable map from an open interval \((-\varepsilon, \varepsilon)\) into \(M\).

**Def:** Given a manifold \(M\) and a point \(p \in M\), the tangent space \(T_p M\) (to \(M\) at \(p\)) is the set of tangent vectors to differential curves s.t. \(\alpha(0) = p\).