Differentiable manifolds:

<u>Idea</u>: Generalize regular surfaces in $\mathbb{R}^3$.

<u>Recall</u>: A regular surface in $\mathbb{R}^3$ is a subset $M \subset \mathbb{R}^3$ ~~s~~ s.t. $\forall p \in M$ $\exists$ a neighborhood V of p in $\mathbb{R}^3$ and a mapping $\varphi: U \longrightarrow V \cap M$ from an open subset U of $\mathbb{R}^2$ s.t.

(a) φ is a differentiable homeomorphism

(b) the differential $(d\varphi)_q : T_q U \longrightarrow T_{\varphi(q)} \mathbb{R}^3$

$$\begin{array}{ccc} T_q U & \longrightarrow & T_{\varphi(q)}\mathbb{R}^3 \\ \cong & & \cong \\ \mathbb{R}^2 & \longrightarrow & \mathbb{R}^3 \end{array}$$

is <u>injective</u> $\forall q \in U$.

<u>Def</u>: $\varphi: U \to V \cap M$ as above is called a coordinate chart on M near p.

Given a regular surface $M \subset \mathbb{R}^3$ and two coordinate charts $\varphi: U \to M$, $\psi: V \to M$ s.t. $W := \varphi(U) \cap \psi(V) \neq \emptyset$ the map $\psi^{-1} \circ \varphi : \varphi^{-1}(W) \to \mathbb{R}^2$ is differentiable

<u>Definition:</u> A differentiable manifold of dimension n is a set M together with the data of a ~~set~~ collection of open subsets $\{V_\alpha\}$ of $\mathbb{R}^n$ and injective maps $\varphi_\alpha : V_\alpha \longrightarrow M$

s.t. :

(1) $\bigcup \varphi_\alpha(V_\alpha) = M$

(2) $\forall\ \alpha, \beta.$ s.t. $\varphi_\alpha(V_\alpha) \cap \varphi_\beta(V_\beta) \neq \emptyset$

$$\varphi_\alpha(V_\alpha) \cap \varphi_\beta(V_\beta) =: W_{\alpha\beta}.$$

$$\begin{tikzcd} \varphi_\beta^{-1}\circ\varphi_\alpha : \varphi_\alpha^{-1}(W_{\alpha\beta}) \arrow[r] \arrow[dr] & \mathbb{R}^n \\ & \varphi_\beta^{-1}(W_{\alpha\beta}) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

is differentiable

<u>Remark:</u> The differentiable structure induces a topology on M:

A subset W of M is open $\Longleftrightarrow$ $\forall \alpha$, $\varphi_\alpha^{-1}(W \cap \varphi(V_\alpha))$ is open

In this topology $\varphi(V_\alpha)$ is open $\forall \alpha$ & φ_α is continuous $\forall \alpha$.

<u>Example:</u> Real projective space
$$\mathbb{P}^n(\mathbb{R}):$$

<u>Def:</u> $\mathbb{P}^n(\mathbb{R}) := \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}^*$.

i.e., $\mathbb{P}^n(\mathbb{R})$ is the set of equivalence classes for the eq. relation: $(x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1})$
$$\Leftrightarrow \exists \lambda \in \mathbb{R}^* \text{ s.t. } y_i = \lambda x_i \quad \forall i$$

equivalently, $\mathbb{P}^n(\mathbb{R})$ is the set of lines through the origin in $\mathbb{R}^{n+1}$.

<u>$\mathbb{P}^n(\mathbb{R})$ is a differentiable manifold:</u>

Define $V_i \subset \mathbb{P}^n$ as the set of equivalence classes $[x_1, \dots, x_{n+1}]$ s.t. $x_i \neq 0$.

note that $\bigcup_{i=1}^{n+1} V_i = \mathbb{P}^n$

also $[x_1, \dots, x_{n+1}] = \left[\frac{x_1}{x_i}, \dots, 1, \dots, \frac{x_{n+1}}{x_i}\right]$

$$\varphi_i : \mathbb{R}^n \longrightarrow \mathbb{P}^n$$
$$(y_1, \dots, y_n) \longmapsto [y_1, \dots, 1, \dots, y_n]$$

φ_i is injective.

to have a differentiable structure, we need $\forall i, j$

$$\varphi_j^{-1} \circ \varphi_i : \varphi_i^{-1}(V_i \cap V_j) \longrightarrow \mathbb{R}^n$$

to be differentiable.

$$V_i \cap V_j = \begin{cases} [x_1, \dots, x_n] & \text{homogeneous coordinates} \\ \text{s.t. } x_i \neq 0, \ x_j \neq 0 \end{cases}$$

$$\varphi_i(y_1, \dots, y_n) = [y_1, \dots, 1, \dots, y_n]$$

$y_j \neq 0$

(WLOG : $i > j$)

$$\varphi_i^{-1}(V_i \cap V_j) = \{(y_1, \dots, y_n) : y_j \neq 0\}$$

$$(\varphi_j^{-1} \circ \varphi_i)(y_1, \dots, y_n) = \varphi_j^{-1}[y_1, \dots, \underset{i\text{-th.}}{1}, \dots, y_n]$$

$$\varphi_j(z_1, \dots, z_n) = [z_1, \dots, \underset{\updownarrow\ j\text{-th}}{1}, \dots, z_n]$$

$$(z_1, \dots, z_n) = \varphi_j^{-1}[z_1, \dots, 1, \dots, z_n]$$

$$[z_1, \dots, 1, \dots, z_n] = \left[\frac{z_1}{z_i}, \dots, \frac{1}{z_i}, \dots, 1, \dots, z_n\right]$$

So $\varphi_j^{-1}\varphi_i(y_1, \dots, y_n) = \varphi_j^{-1}[y_1, \dots, 1, \dots, y_n]$

$$= \left(\frac{y_1}{y_i}, \dots, \frac{y_n}{y_i}\right)$$

$$= \left(\frac{y_1}{y_i}, \dots, \frac{y_{j-1}}{y_i}, \frac{y_{j+1}}{y_i}, \dots, \frac{1}{y_i}, \dots, \frac{y_n}{y_i}\right)$$

is differentiable where $y_i \neq 0$

<u>Differentiable maps</u>:

<u>Def</u>: Given two manifolds M_1, M_2, a map $\varphi : M_1 \longrightarrow M_2$ is differentiable if $\forall p \in M_1$, $\exists$ $V \subset \mathbb{R}^n$

$g : V \longrightarrow M_2$ a coordinate chart s.t. $g(V) \ni \varphi(p)$

$\exists$ $f : U \longrightarrow M_1$, $U \subset \mathbb{R}^m$ coordinate chart of M_1 s.t. $p \in f(U)$

~~$\exists\ f: U \longrightarrow M_1$, ~~st~~ $V \subset \mathbb{R}^m$~~

~~coordinate chart of M_1~~

and ~~st~~ $\varphi(f(U)) \subset g(V)$

and $(g^{-1}\varphi f): U \longrightarrow V \subset \mathbb{R}^n$
$U \cap \mathbb{R}^m$

is differentiable.

<u>Note</u>: With this definition any coordinate chart $\varphi_\alpha : V_\alpha \to M$ is differentiable.

<u>Def</u>: A differentiable curve in M is a differentiable map from an open interval $(-\varepsilon, \varepsilon)$ into M.

<u>Def</u>: Given a manifold M and a point $p \in M$, the tangent space $T_p M$ (to M at p) is the set of tangent vectors to differential curves s.t. $\alpha(0) = p$.