Def: Suppose given a manifold $M$ and a differentiable curve $\alpha$ in $M$ s.t. $\alpha(0) = p$. Let $D \alpha$ be the set of differentiable functions from a neighborhood of $p \in M$ to $\mathbb{R}$. The tangent vector at $p$ to $\alpha$ is the operator on $D$ which to a function $f$ associates:

$$\frac{d}{dt}(f \circ \alpha) \bigg|_{t=0}$$

In other words: $\alpha'(0)(f) = \frac{d}{dt}(f \circ \alpha) /_{t=0}$

Example: $M = \mathbb{R}^n$, $\alpha : (-\varepsilon, \varepsilon) \to M$

$\gamma = 0 = (0, \ldots, 0)$, $\alpha(0) = 0$.

$\alpha(t) = (x_1(t), \ldots, x_n(t))$, $t \in (-\varepsilon, \varepsilon)$

$f : U \to \mathbb{R}$, $0 \in U \subset \mathbb{R}^n$

$q \in U$, $f(q) = f(x_1, \ldots, x_n)$, $q = (x_1, \ldots, x_n)$.
\[ x'(0)(f) = \frac{d}{dt} (f(x(t))) \bigg|_{t=0} = \frac{d}{dt} \left( f(x_1(t), \ldots, x_n(t)) \right) \bigg|_{t=0} \]

Chain rule:
\[ \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} (x_i(0)) \]

Directional derivative of \( f \) in the direction of \( (x'_1(0), \ldots, x'_n(0)) \)

On a general manifold \( M \):
\[ \forall \, \mathbf{f} \in M \exists \, \psi : U \subset M \ni \mathbf{f} \in U \subset \mathbb{R}^n \text{ coordinate chart.} \]

A tangent vector is the velocity vector of some curve \( \mathbf{x} : (-\varepsilon, \varepsilon) \rightarrow M \) s.t. \( \mathbf{x}(0) = \mathbf{f} \).

Take \( \varepsilon \) small enough s.t. the image of \( \mathbf{x} \) is contained in \( U \).
\[ \forall \psi \in \mathcal{F}(U) \]
\[ \psi_*(\psi^{-1}(q)) = (x_1, \ldots, x_n) \in \mathbb{R}^n \]

\( f : \text{some neighborhood of } f \rightarrow \mathbb{R} \)

shrink \( U \) if necessary so that

\[ f : \psi(U) \rightarrow \mathbb{R} \]

\[ \alpha'(0)(f) = \frac{d}{dt} (f \circ \alpha)(t) \bigg|_{t=0} \]

\[ f(\alpha(t)) = f(\psi(x_1(t), \ldots, x_n(t))) \]

\[ \alpha(t) \in \psi(U) \]

\[ \psi^{-1}(\alpha(t)) = (x_1(t), \ldots, x_n(t)) \]

\[ f(\alpha(t)) = f(\psi(\psi^{-1}(\alpha(t)))) \]

\[ = f\left(\psi(x_1(t), \ldots, x_n(t))\right) \]

\[ = (f \circ \psi)(x_1(t), \ldots, x_n(t)) \]

\[ \frac{d}{dt} (f \circ \alpha(t)) \bigg|_{t=0} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (f \circ \psi)(0) x_i'(0) \]

So \( \alpha'(0) \) has the coordinates \((x_1'(0), \ldots, x_n'(0))\) in the chart \( \psi : U \rightarrow \mathbb{M} \).
\[ \frac{\partial}{\partial x^i} \] is the velocity vector of the coordinate curve \( \lambda(t) = \left(0, \ldots, t, 0, \ldots\right) \) \( i \text{-th} \)

Then \( \lambda'(0) = \sum_{i=1}^{n} x_i'(0) \frac{\partial}{\partial x^i} \)

for arbitrary \( \lambda \).

This description shows that \( T_p M \) is a vector space \( \cong \mathbb{R}^n \).

In each chart \( \varphi : U \to M \), the vectors \( \frac{\partial}{\partial x^i} \) form a basis of \( T_p M \).

**Differentials of maps:**

**Def:** Suppose given two manifolds \( M_1, M_2 \) and a differentiable map \( a : M_1 \to M_2 \).

Given \( f \in C^\infty(M_1) \), the differential of \( a \) at \( p \) is a linear map \( da : T_p M_1 \to T_{a(p)} M_2 \) defined as follows:
Given a function \( f \) defined on a neighborhood of \( q(h) \) in \( M_2 \),

\[
(d\alpha)(f)(x) := \left( \frac{d}{dt} \right)(f \circ \alpha)(\alpha(t)) \bigg|_{t=0}
\]

In coordinate charts:

\( \alpha : M_1 \rightarrow M_2 \)  \( \dim M_1 = m, \dim M_2 = n \)

\( \psi \in V \subseteq \mathbb{R}^m \)  \( (x_1, \ldots, x_m) \)

\( \alpha(t) \in V \subseteq \mathbb{R}^n \)  \( (y_1, \ldots, y_n) \)

\( T_{\alpha(q)}M_1 = \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m} \right\rangle \)

\( T_{\alpha(q)}M_2 = \left\langle \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \right\rangle \)

\( \nu \in T_{\alpha(q)}M_1 \), \( \alpha : (-\varepsilon, \varepsilon) \rightarrow M_1 \),

s.t.  \( \alpha(0) = q \),  \( \alpha'(0) = \nu \)

\( \psi : U \rightarrow M_1 \),  \( \psi : V \rightarrow M_2 \)
\[
\begin{align*}
\frac{d}{dt} f^a(x) (t) \bigg|_{t=0} &= \sum_{j=1}^n \left( \frac{\partial}{\partial y_j} \right) f^a(0) \sum_{i=1}^m \frac{\partial y_j}{\partial x_i} \left( \frac{\partial}{\partial x_i} \right) x^i(0) \\
\frac{d}{dt} \left( \frac{\partial}{\partial y_j} \right) f^a(0) &= \sum_{i=1}^m \frac{\partial y_j}{\partial x_i} \left( \frac{\partial}{\partial x_i} \right) x^i(0) \\
\end{align*}
\]

So \( (da)(x^i(0), \ldots, x^m(0)) = \sum_{j=1}^n \left( \frac{\partial y_j}{\partial x_i} \right) \frac{\partial y_j}{\partial x_i} \left( \frac{\partial}{\partial x_i} \right) x^i(0) \). 

So \( da : T^t M_1 \rightarrow T_a(t) M_2 \) has matrix \( \left( \frac{\partial y_j}{\partial x_i} \right)_{1 \leq i \leq m, 1 \leq j \leq n} \) in the bases \( \left\{ \frac{\partial}{\partial x_i} \right\} \) and \( \left\{ \frac{\partial}{\partial y_j} \right\} \).
\[ f : \mathcal{V} \rightarrow \mathbb{R}. \]

\[ (\partial a)(v) = (\partial a) (\alpha'(0)) \]

\[ (\partial a)(v) (f) = \frac{d}{dt} (f \circ a \circ \alpha)(t) \bigg|_{t=0} \]

\[ \psi'^{-1}(U) \xrightarrow{\psi} V \xrightarrow{\psi'^{-1}} \text{shrink } U \text{ if necessary} \]

\[ \begin{align*}
(1, \ldots, x_n) &\mapsto \psi(1, \ldots, x_n) \\
\quad &\mapsto a(\psi(1, \ldots, x_n)) \\
\quad &\mapsto \psi^{-1}(a(\psi(1, \ldots, x_n))) \\
\quad &\mapsto \begin{pmatrix}
\psi^{-1}(1), \ldots, \\
\vdots \\
\psi^{-1}(n)
\end{pmatrix}
\end{align*} \]

\[ f \circ a = f \psi \psi'^{-1} a \]

\[ = (f \psi)(1, \ldots, x_n) \]

\[ = (f \psi)(\psi(1, \ldots, x_n), \ldots, \psi(n, \ldots, x_n)) \]

\[ (f \circ a \circ \alpha)(t) = (f \psi)(\psi(1, \alpha(t), \ldots, x_m(t)), \ldots, \psi(n, \alpha(t), \ldots, x_m(t))) \]
application: \( \varphi_1 : V_1 \to M \) \( \varphi_2 : V_2 \to M \)

\[ M_1 = \varphi_1(U_1) \quad M_2 = \varphi_2(U_2) \]

\( \alpha = \varphi_2^{-1} \circ \varphi_1 : V_1 \to V_2. \)

**Def:** \( \alpha : M_1 \to M_2 \) is a diffeomorphism if \( \alpha \) is differential, \( \alpha \) is a bijection and \( \alpha^{-1} \) is also differential. The map \( \alpha \) is a local diffeomorphism if \( \forall \, p \in M_1 \), \( \exists \) neighborhoods \( U \) of \( p \) in \( M_1 \), \( V \) of \( \alpha(p) \) in \( M_2 \) s.t. \( \alpha|_U : U \to V \) is a diffeomorphism.

**Theorem (2.10):** If \( \varphi : M_1 \to M_2 \) is a differentiable map, then \( \varphi \) is a local diffeomorphism iff
A \phi \in \mathcal{M}, (\circ \phi) \phi \text{ is an isom.}

\phi \text{ is a local diffeomorphism iff \phi is injective and a local diffeo-
morphism.}