Def: Suppose given a manifold M and a differentiable curve α in M s.t. $\alpha(0) = p$. Let $\mathcal{D}$ be the set of differentiable functions from a neighborhood of $p \in M$ to $\mathbb{R}$.

The tangent vector at p to α is the operator on $\mathcal{D}$ which to a function f associates.

$$\left.\frac{d}{dt}(f \circ \alpha)\right|_{t=0}$$

In other words: $\alpha'(0)(f) = \left.\frac{d}{dt}(f\circ\alpha)\right|_{t=0}$

Example: $M = \mathbb{R}^n$ $\quad \alpha : (-\varepsilon, \varepsilon) \longrightarrow M$

$p = 0 = (0, \cdots, 0) \qquad \alpha(0) = 0.$

$\alpha(t) = (x_1(t), \dots, x_n(t)) \qquad t \in (-\varepsilon, \varepsilon)$

$f : U \longrightarrow \mathbb{R} \qquad 0 \in U \subset \mathbb{R}^n$

$q \in U \qquad f(q) = f(x_1, \dots, x_n), \quad q = (x_1, \dots, x_n)$

$$\alpha'(0)(f) = \frac{d}{dt}(f\circ\alpha)\Big|_{t=0}.$$

$$= \frac{d}{dt}\left(f(x_1(t), \dots, x_n(t))\right)\Big|_{t=0}$$

chain rule $\Rightarrow$

$$= \sum_{i=1}^{n} \frac{\partial f(0)}{\partial x_i} \cdot x_i'(0)$$

= directional derivative of f in the direction of

$(x_1'(0), \dots, x_n'(0))$

traditional velocity vector.

On a general manifold M:

$p \in M$ $\quad \exists\ \varphi : U \hookrightarrow M$

s.t. $p \in U \subset \mathbb{R}^n$ coordinate chart.

a tangent vector is the velocity vector of some curve. $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$

s.t. $\alpha(0) = p$.

take ε small enough s.t. the image of α is contained in U

$\forall\ q \in \varphi(U)$

$$\varphi^{-1}(q) = (x_1, \dots, x_n) \in \mathbb{R}^n$$

f: some neighborhood of $p \to \mathbb{R}$

shrink U if necessary so to

$$f : \varphi(U) \longrightarrow \mathbb{R}$$

$$\alpha'(0)(f) = \frac{d}{dt}(f \circ \alpha)(t)\Big|_{t=0}$$

$$\cancel{f(\alpha(t)) = f(\varphi(x_1, \dots, x_n)}$$

$$\alpha(t) \in \varphi(U)$$

$$\varphi^{-1}(\alpha(t)) = (x_1(t), \dots, x_n(t))$$

$$f(\alpha(t)) = f(\varphi(\varphi^{-1}(\alpha(t))))$$

$$= f(\varphi(x_1(t), \dots, x_n(t)))$$

$$= (f \circ \varphi)(x_1(t), \dots, x_n(t))$$

$$\frac{d}{dt}(f \circ \alpha(t))\Big|_{t=0} = \sum_{i=1}^{n} \frac{\partial (f \circ \varphi)}{\partial x_i}(0)\, x_i'(0)$$

So $\alpha'(0)$ has the coordinates

$(x_1'(0), \dots, x_n'(0))$ in the chart $\varphi : U \to M$

$\frac{\partial}{\partial x_i}$ is the velocity vector of the coordinate curve $\alpha(t) = \varphi(0, \ldots, t, 0, \ldots)$
↖ i-th

then $\alpha'(0) = \sum_{i=1}^{n} x_i'(0) \frac{\partial}{\partial x_i}$

for arbitrary α.

This description shows that $T_p M$ is a vector space $\approx \mathbb{R}^n$.

In each chart $\varphi: U \to M$, the vectors $\frac{\partial}{\partial x_i}$ form a basis of $T_p M$.

Differentials of maps:

<u>Def</u>: Suppose given two manifolds M_1, M_2 and a differentiable map $a: M_1 \longrightarrow M_2$.

given $p \in M_1$, the differential of a at p is a linear ~~map~~ map $\underline{da_p}: T_p M_1 \longrightarrow T_{\varphi(p)} M_2$ defined as follows:

$v \in T_p M_1$, $\exists\ \alpha : (-\varepsilon, \varepsilon) \to M_1$

s.t. $\alpha'(0) = v$

given a function f defined on a neighborhood of $a(p)$ in M_2

$$(da)_p(v)(f) := (da)_p(\alpha'(0))(f) := \frac{d}{dt}(f \circ a)(\alpha(t))\Big|_{t=0}$$

<u>In coordinate charts:</u>

$a : M_1 \to M_2$ $\quad \dim M_1 = m, \dim M_2 = n$

$\overset{\Psi}{p} \in U \subset \mathbb{R}^m$ $\quad (x_1, \dots, x_m)$

$a(p) \in V \subset \mathbb{R}^n$ $(y_1, \dots, y_n)$

$$T_p M_1 = \left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \right\rangle$$

$$T_{a(p)} M_2 = \left\langle \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \right\rangle$$

$v \in T_p M_1$ $\quad \alpha : (-\varepsilon, \varepsilon) \to M_1$

s.t. $\alpha(0) = p$ $\quad \alpha'(0) = v$

$\varphi : U \to M_1$, $\psi : V \to M_2$

$$\frac{d}{dt}(f\circ a\circ\alpha)(t)\Big|_{t=0} =$$

$$= \sum_{j=1}^{n} \frac{\partial}{\partial y_j}(f\psi)(0) \sum_{i=1}^{m} \frac{\partial y_j}{\partial x_i}(0)\, x_i'(0)$$

$$\cancel{= \left(\frac{\partial y_j}{\partial x_i}(0)\right)_{\substack{1\le i\le m\\ 1\le j\le n}}}$$

So $(da)(x_1'(0), \dots, x_m'(0)) =$

$$= \sum_{j=1}^{n}\left(\sum_{i=1}^{m} \frac{\partial y_j}{\partial x_i}(0)\, x_i'(0)\right)\frac{\partial}{\partial y_j}.$$

So $da : T_p M_1 \longrightarrow T_{a(p)} M_2$

has matrix $\left(\frac{\partial y_j}{\partial x_i}(0)\right)_{\substack{1\le i\le m\\ 1\le j\le n}}$

in the bases $\left\{\frac{\partial}{\partial x_i}\right\}$

and $\left\{\frac{\partial}{\partial y_j}\right\}$.

$$f : \psi(V) \longrightarrow \mathbb{R}.$$

$$(da)(v) = (da)(\alpha'(0))$$

$$(da)(v)(f) = \frac{d}{dt}(f \circ a \circ \alpha)(t)\Big|_{t=0}$$

$$\begin{tikzcd} U \arrow[r] \arrow[d, shift left, "\varphi"] & V \arrow[d, shift right, "\psi"'] \\ M_1 \arrow[r, "a"'] \arrow[u, bend left, "\varphi^{-1}"] & M_2 \arrow[u, shift right, "\psi^{-1}"'] \end{tikzcd}$$

shrink U if necessary

$$(x_1, \ldots, x_m) \longmapsto \varphi(x_1, \ldots, x_m)$$
$$\longmapsto a(\varphi(x_1, \ldots, x_m))$$
$$\longmapsto \psi^{-1}(a(\varphi(x_1, \ldots, x_m)))$$
$$\| $$
$$(y_1(x_1, \ldots, x_m), \ldots, y_n(x_1, \ldots$$

$$f \circ a = f\psi\psi^{-1}a$$
$$= (f\psi)(y_1, \ldots, y_n)$$
$$= (f\psi)(y_1(x_1, \ldots, x_m), \ldots, y_n(x_1, \ldots, x_m))$$

$$(f \circ a \circ \alpha)(t) = (f\psi)(y_1(x_1(t), \ldots, x_m(t)), \ldots\ldots)$$

application:

$$\varphi_1 : U_1 \longrightarrow M \qquad \varphi_2 : U_2 \longrightarrow M$$

$$M_1 = (U_1) \qquad M_2 = (U_2)$$

$$a = \varphi_2^{-1}\,\varphi_1 : U_1 \longrightarrow U_2 .$$

Def: $a : M_1 \longrightarrow M_2$ is a diffeomorphism if a is differentiable, a is a bijection and a^{-1} is also differentiable. The map a is a local diffeomorphism if $\forall\, p \in M_1$ $\exists$ neighborhoods U of p in M_1, V of $a(p)$ in M_2 s.t.

$$a\big|_U : U \longrightarrow V \text{ is a diffeomorphism.}$$

Theorem (2.10): If $\varphi : M_1 \longrightarrow M_2$ is a differentiable map, then φ is a local diffeomorphism iff

$\forall\ p \in M$, $(d\varphi)_p$ is an isom.

φ is a ~~to~~ diffeomorphism iff φ is injective and a local diffeomorphism.