$M$ a Riemannian manifold.

$\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$

$(X, Y) \mapsto \nabla_X Y.$

A connection,
(linear in $X$, satisfies the Leibniz rule in $Y$

$\nabla$ is compatible with the metric

when $\forall \ X, Y, Z \in \mathfrak{X}(M)$

$\nabla \langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle$

equivalently $\forall$ curve $\alpha: I \to M$

$\frac{d}{dt} \langle X, Y \rangle = \langle \frac{DX}{dt}, Y \rangle + \langle X, \frac{DY}{dt} \rangle$

Remark: if $X, Y$ are parallel on $\alpha$,
then $\langle X, Y \rangle$ is constant on $\alpha$.

Also, if $X$ is parallel, then $X$ has constant length $|X| = \sqrt{\langle X, X \rangle}$

$\frac{d}{dt} |X| = \frac{d}{dt} \sqrt{\langle X, X \rangle} = \frac{1}{2} \left( \frac{d}{dt} \langle X, X \rangle \right) \frac{1}{|X|}$
Remark: Given a symmetric inner product $\langle , \rangle$ on a vector space $V$, $\langle , \rangle$ is determined by its values $\langle X, X \rangle$ for $X \in V$ (this is the associated quadratic form).

$\langle X + Y, X + Y \rangle = \langle X, X \rangle + 2 \langle X, Y \rangle + \langle Y, Y \rangle$ 

$\Rightarrow \langle X, Y \rangle = \frac{1}{2}(\langle X + Y, X + Y \rangle - (\langle X, X \rangle + \langle Y, Y \rangle))$

Def: $\nabla$ is symmetric if $\forall X, Y$

$\nabla_X Y - \nabla_Y X = [X, Y]$

compute: equivalent to $\Gamma_{ij}^k = \Gamma_{ji}^k$

$\forall i, j, k$.

Def: $\nabla$ is Riemannian or Levi-Civita if $\nabla$ is symmetric and compatible with the metric.

Theorem: Any Riemannian manifold has a unique Riemannian connection.
Proof: Given $x, y, z \in \mathcal{C}(M)$

\[
X \langle y, z \rangle = \langle \nabla_x y, z \rangle + \langle y, \nabla_x z \rangle \\
y \langle z, x \rangle = \langle \nabla_y z, x \rangle + \langle z, \nabla_y x \rangle \\
z \langle x, y \rangle = \langle \nabla_z x, y \rangle + \langle x, \nabla_z y \rangle
\]

Add the first two and subtract the third:

\[
X \langle y, z \rangle + y \langle z, x \rangle - z \langle x, y \rangle = \\
\langle \nabla_x y \nabla_z + \nabla_y x, z \rangle + \langle \nabla_z x - \nabla_z x, y \rangle \\
+ \langle \nabla_y z - \nabla_z y, x \rangle
\]

\[
= \langle [x, y] + 2 \nabla_y x, z \rangle + \langle [x, z], y \rangle \\
+ \langle [y, z], x \rangle
\]

\[
= \langle [x, y], z \rangle + \langle [y, z], x \rangle + \langle [x, z], y \rangle \\
+ 2 \langle \nabla_y x, z \rangle
\]

\[
= \langle \nabla_y, x, z \rangle = \frac{1}{2} \left( X \langle y, z \rangle + y \langle z, x \rangle + z \langle x, y \rangle - 2 \langle x, y \rangle \right)
\]

The formula determines $\nabla_y x$.
can choose $Z(x)$ to be a vector of an orthonormal basis of $T_x M$.

The formula proves uniqueness of a Riemannian connection.

To prove existence, use the formula as a definition locally near every point of $M$. Use uniqueness on intersections of neighborhoods to show $\nabla_X Y$ is well-defined globally.

\[ \nabla_{X} Y = \frac{1}{2} \left( \frac{d}{dt} \left[ X(Y) \right] - Y(X) \right) \]

\textbf{Geodesics:}

**Def:** A curve $\gamma : I \rightarrow M$ is a geodesic at a point $t_0 \in I$ if \( \frac{d}{dt} \left( \frac{d\gamma}{dt} \right) \bigg|_{t=t_0} = 0 \).

We say $\gamma$ is a geodesic if it is a geodesic at every point of $I$.

If $[a, b] \subset I$ and $\gamma$ is a geodesic, then $\gamma([a, b])$ is a geodesic segment from $\gamma(a)$ to $\gamma(b)$. 
Remarks: the velocity vector of a geodesic has constant length:
\[
\frac{d}{dt} \langle \frac{dr}{dt}, \frac{ds}{dt} \rangle = 2 \langle \frac{D}{dt} (\frac{ds}{dt}), \frac{ds}{dt} \rangle = 0
\]
This means: the parametrization is not arbitrary. \((s(I) \text{ is not a point})\)
\[
c := \left| \frac{ds}{dt} \right| \neq 0 \quad s(t) = \int_0^t \left| \frac{dr}{dt} \right| dt
\]
length of \(s(t)\):
\[
s(t) = c(t-t_0)
\]
So parameter is proportional to the length.
Special case \(c=1\): \(s\) is parametrized by arc length or \(s\) is normalized on \(s\) has unit speed.

Recall \(TM = \bigcup_{t \in M} T_x M\)
\(\pi: U \to M\) coordinate chart
\( x^i(t) \rightarrow t \in \mathcal{M} \)

\( = (x_1(t), \ldots, x_n(t)) \).

\( T_p U = T_p M = \left< \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right> \)

\( \forall \in U \rightarrow T_p M = \bigoplus_{i=1}^n v_i \frac{\partial}{\partial x_i} \).

So we obtain

\( \ast : T \mathcal{U} = U \times \mathbb{R}^n \longrightarrow T \mathcal{M} \).

\( (x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n) \mapsto (\ast(x_1, x_n), \Sigma v_i \frac{\partial}{\partial x_i}) \)

a coordinate chart for \( T \mathcal{M} \).

Any curve \( \gamma : I \rightarrow \mathcal{M} \) gives

a curve in \( T \mathcal{M} \) via

\( t \rightarrow (\gamma(t), \frac{d\gamma}{dt}) \)

Question: when is \( \gamma \) a geodesic in terms of these coordinates?

\( x^i(\gamma(t)) = (x_1(t), \ldots, x_n(t)) \)

\( \frac{d\gamma}{dt} = \sum_{i=1}^n \frac{dx_i}{dt} \frac{\partial}{\partial x_i} \).
\[
\frac{D}{dt} \left( \frac{d\mathbf{x}}{dt} \right) = \sum_{i=1}^{n} \frac{d^2x_i}{dt^2} + \sum_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt} \Gamma_{ij}^k \]  

\forall k \quad \frac{d^2x_k}{dt^2} + \sum_{ij} \frac{dx_i}{dt} \frac{dx_j}{dt} \Gamma_{ij}^k = 0. \forall k.

Second order diff. eq.

In TM it becomes first order:

\begin{align*}
\left\{ \begin{array}{l}
\mathbf{v}_i = \frac{dx_i}{dt} \quad \forall i, \\
\frac{d\mathbf{v}_k}{dt} + \sum_{ij} \mathbf{v}_i \mathbf{v}_j \Gamma_{ij}^k = 0. \forall k.
\end{array} \right.
\end{align*}

This shows that geodesics exist and are unique (given initial conditions: point + velocity vector locally).
Theorem: (Lemma 3.4) \( \exists ! \) vector field \( \xi \) on \( TM \) whose trajectories are of the form \( (\gamma(t), \frac{d\gamma}{dt}) \) where \( \gamma \) is a geodesic.

Proof: uniqueness: if it exists, then \( \sum v_i = \frac{dx_i}{dt} + i \)

\[
\frac{dv_k}{dt} + \sum v_i v_j \Gamma^k_{ij} = 0 \quad \forall k.
\]

Coordinates of \( G \) are \( (v_i, \frac{dv_i}{dt}) \).

Uniqueness of solutions of systems of ODE gives uniqueness of the coordinates of \( G \).

Existence: the solution to the equations exist \( \implies \) \( G \) locally.

We use uniqueness in intersections of open sets to prove global existence.