M a Riemannian manifold.

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$$
$$(X, Y) \longmapsto \nabla_X Y.$$

a connection.

(linear in X, satisfies the Leibnitz rule in Y)

∇ is compatible with the metric when $\forall$ $X, Y, Z \in \mathcal{X}(M)$

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$

equivalently $\forall$ curve $\alpha : I \to M$

$$\frac{d}{dt}\langle X, Y\rangle = \left\langle \frac{DX}{dt}, Y\right\rangle + \left\langle X, \frac{DY}{dt}\right\rangle$$

<u>Remark</u>: If X, Y are parallel on α, then $\langle X, Y\rangle$ is constant on α.

Also, if X is parallel, then X has constant length. $|X| = \sqrt{\langle X, X\rangle}$

$$\frac{d}{dt}|X| = \frac{d}{dt}\sqrt{\langle X, X\rangle} = \frac{1}{2}\left(\frac{d}{dt}\langle X, X\rangle\right)\frac{1}{|X|}$$

Remark: Given a symmetric inner product $\langle\,,\,\rangle$ on a vector space V, $\langle\,,\,\rangle$ is determined by its values $\langle X, X\rangle$ for $X \in V$ (this is the associated quadratic form).

$$\langle X+Y, X+Y\rangle = \langle X, X\rangle + 2\langle X, Y\rangle + \langle Y, Y\rangle$$

$$\Rightarrow \langle X, Y\rangle = \frac{1}{2}\Big(\langle X+Y, X+Y\rangle - (\langle X, X\rangle + \langle Y, Y\rangle)\Big)$$

Def: ∇ is symmetric if $\forall\, X, Y$

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

compute: equivalent to $\Gamma_{ij}^k = \Gamma_{ji}^k$ $\forall\, i, j, k$.

Def: ∇ is Riemannian or Levi-Civita if ∇ is symmetric and compatible with the metric.

Theorem: Any Riemannian manifold has a unique Riemannian connection.

Proof: given $X, Y, Z \in \mathfrak{X}(M)$

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

$$Y\langle Z, X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle$$

$$Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$

add the first two and subtract the third:

$$X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle =$$

$$\langle \nabla_X Y + \nabla_Y X, Z\rangle + \langle \nabla_X Z - \nabla_Z X, Y\rangle$$

$$+ \langle \nabla_Y Z - \nabla_Z Y, X\rangle$$

$$= \langle [X, Y] + 2\nabla_Y X, Z\rangle + \langle [X, Z], Y\rangle$$

$$+ \langle [Y, Z], X\rangle$$

$$= \langle [X, Y], Z\rangle + \langle [Y, Z], X\rangle + \langle [X, Z], Y\rangle$$

$$+ 2\langle \nabla_Y X, Z\rangle$$

$$\Rightarrow \langle \nabla_Y X, Z\rangle = \frac{1}{2}\left\{ \begin{array}{l} X\langle Y, Z\rangle + Y\langle Z, X\rangle + \\ -Z\langle X, Y\rangle \\ -\langle [X, Y], Z\rangle \\ -\langle [Y, Z], X\rangle \\ +\langle [Z, X], Y\rangle \end{array} \right.$$

the formula determines $\nabla_{Y_p} X$: we can choose $Z(p)$ to be a vector of an orthonormal basis of $T_p M$.

$\rightarrow$ formula proves uniqueness of a Riemannian connection.

To prove existence, use the formula as a definition locally near every point of M. Use uniqueness on intersection of neighborhoods to ~~show~~ show $\nabla_Y X$ is well-defined globally.

$\square$.

Geodesics:

Def: A curve $\gamma : I \longrightarrow M$ is a geodesic at a point $t_0 \in I$ if

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)\Big|_{t=t_0} = 0 .$$

We say γ is a geodesic if it is a geodesic at every point of I.

If $[a,b] \subset I$ and γ is a geodesic, then $\gamma([a,b])$ is a geodesic segment from $\gamma(a)$ to $\gamma(b)$.

Remarks: the velocity vector of a geodesic has constant length:

$$\frac{d}{dt}\left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right\rangle = 2\left\langle \frac{D}{dt}\left(\frac{d\gamma}{dt}\right), \frac{d\gamma}{dt}\right\rangle$$

$$= 0.$$

This means: the parametrization is not arbitrary. ($\gamma(I)$ is NOT a point)

$$c := \left|\frac{d\gamma}{dt}\right| \neq 0$$

length of $\gamma(t)$: $s(t) = \int_{t_0}^{t} \left|\frac{d\gamma}{dt}\right| dt$

$$s(t) = c(t - t_0)$$

So parameter is proportional to the length.

Special case $c = 1$: γ is parametrized by arc length or γ is normalized or γ has unit speed.

Recall $TM = \coprod_{p\in M} T_p M$

$x : U \longrightarrow M$ coordinate chart
$\quad \subset \mathbb{R}^n$.

$$x^{-1}(p) \longleftarrow\!\shortmid\, p \in M$$
$$= (x_1(p), \ldots, x_n(p)).$$
$$T_p U = T_p M = \left\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\rangle$$
$$v \in T_p U = T_p M = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}$$

So we obtain

$$X : TU = U \times \mathbb{R}^n \longrightarrow TM.$$
$$(x_1, \ldots, x_n, v_1, \ldots, v_n) \longmapsto \left(x(x_1, \ldots, x_n), \sum v_i \frac{\partial}{\partial x_i}\right)$$

a coordinate chart for TM.

Any curve $\gamma : I \to M$ gives a curve in TM via

$$t \longmapsto \left(\gamma(t), \frac{d\gamma}{dt}\right)$$

<u>Question</u>: when is γ a geodesic in terms of these coordinates?

$$x^{-1}(\gamma(t)) = (x_1(t), \ldots, x_n(t))$$
$$\frac{d\gamma}{dt} = \sum_{i=1}^{n} \frac{dx_i}{dt} \frac{\partial}{\partial x_i}$$

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \sum_{i=1}^{n} \frac{d^2 x_i}{dt^2}\frac{\partial}{\partial x_i} + \sum_{i,j} \frac{dx_i}{dt}\frac{dx_j}{dt} \nabla_{\frac{\partial}{\partial x_j}}\left(\frac{\partial}{\partial x_i}\right)$$

$$= \sum_{i=1}^{n} \frac{d^2 x_i}{dt^2}\frac{\partial}{\partial x_i} + \sum_{i,j,k} \frac{dx_i}{dt}\frac{dx_j}{dt}\Gamma_{ij}^{k}\frac{\partial}{\partial x_k}$$

$$\forall k \qquad \frac{d^2 x_k}{dt^2} + \sum_{i,j} \frac{dx_i}{dt}\frac{dx_j}{dt}\Gamma_{ij}^{k} = 0.$$

second order diff. eq.

but on TM it becomes first order: $(x_1, \ldots, x_n, v_1, \ldots, v_n)$ vector $\sum v_i \frac{\partial}{\partial x_i}$

$$\begin{cases} v_i = \dfrac{dx_i}{dt} & \forall i \\[2ex] \dfrac{dv_k}{dt} + \displaystyle\sum_{i,j} v_i v_j \Gamma_{ij}^{k} = 0 & \forall k. \end{cases}$$

This shows that locally geodesics ~~are~~ exist and are unique (given initial conditions: point + velocity vector)

Theorem: (Lemma 3.4) $\exists!$ vector field G on TM whose trajectories are of the form $\left(\gamma(t), \frac{d\gamma}{dt}\right)$ where γ is a geodesic.

Proof: uniqueness: if it exists, then

$$\begin{cases} v_i = \dfrac{dx_i}{dt} & \forall i \\ \dfrac{dv_k}{dt} + \displaystyle\sum_{i,j} v_i v_j \Gamma_{ij}^k = 0 & \forall k. \end{cases}$$

coordinates of G are $\left(v_i, \frac{dv_i}{dt}\right)$

~~...~~ uniqueness of solutions of systems of D.E. gives uniqueness of the coordinates of G.

existence: the solution to the equations exists $\Rightarrow$ G locally.

~~...~~ use uniqueness in intersections of open sets to prove global existence. $\square$.