Def: G is the geodesic field.
Its flow is the geodesic flow.

Apply the theorem from diff. eq. (2.2 in book)
to G.

$\Rightarrow$ $\cancel{\exists}$ $\forall$ $(p, \cancel{0}) \in TM$
$\exists$ $\mathcal{U} \subset TM$ neighborhood of $(p, 0)$
and a positive real number δ.
and a differentiable map

$$\gamma :]-\delta, \delta[\times \mathcal{U} \longrightarrow TM$$
$$(t, q, v) \longmapsto \gamma(t, q, v)$$

s.t. the map $t \longmapsto \gamma(t, q, v)$
$(\forall\ (q, v) \in \mathcal{U})$
is the unique (geodesic) curve ~~path~~ s.t.
$\gamma(0, q, v) = (q, v)$
and the velocity vector of $\gamma_{(q,v)}$
at 0 is $G(q, v)$.

Recall; if $M = \bigcup \underbrace{U_\alpha}_{\text{coord. neighb.}}$

then $TM = \bigcup U_\alpha \times \mathbb{R}^n$

$= \bigcup TU_\alpha$.

Because $U_\alpha \times \mathbb{R}^n$ has the product topology, $\exists$ V open neighborhood of $p \in M$ and $\exists$ $\varepsilon > 0$ s.t. $V \times B_\varepsilon(0) \subset \mathcal{U}$.

$B_\varepsilon(0)$ ← open ball of radius ε centered at 0

Denote $\mathcal{V}_\varepsilon := V \times B_\varepsilon(0)$.

So: $\boxed{\gamma :]-\delta, \delta[\times \mathcal{V}_\varepsilon \longrightarrow M.}$

<u>Homogeneity lemma</u>: Suppose that the geodesic $\gamma(t, q, v)$ is defined on $]-\delta, \delta[$, then for all $a > 0$, the geodesic $\gamma(t, q, av)$ is defined on $]-\frac{\delta}{a}, \frac{\delta}{a}[$

and furthermore:

$$\gamma(at, q, v) = \gamma(t, q, av)$$

$$\forall \quad (t, q, v) \in]-\delta, \delta[\times \mathcal{V}_\varepsilon .$$

<u>Proof</u>: denote $h(t) := \pi\gamma(at, q, v)$.

well-defined for $(t, q, v) \in]-\frac{\delta}{a}, \frac{\delta}{a}[\times \mathcal{V}_\varepsilon$

use uniqueness of geodesics:

$$h(0) = \pi\gamma(0, q, v) = \cancel{(q,v)}\, q$$

$$h'(0) = \frac{d}{dt} \gamma(at, q, v) \Big|_{t=0}.$$

$$= a \frac{d\gamma}{dt}(0, q, v) = av.$$

verify h is a geodesic:

$$\frac{D}{dt} h'(t) = \frac{D}{dt}(a\gamma'(t)) = \nabla_{h'(t)}(h'(t))$$

$$= \nabla_{a\gamma'(t)}(a\gamma'(t)) = a^2 \nabla_{\gamma'(t)} \gamma'(t)$$

$$= 0$$

$$\begin{array}{cccc} \pi : & TM & \longrightarrow & M. \\ & \cup & & \\ & U_\alpha \times \mathbb{R}^n & \longrightarrow & U_\alpha \end{array}$$

So we have that h is the unique geodesic taking the value q at 0 and with velocity vector av at 0.

Now lift $h(t)$ to TM:

$$t \longmapsto (h(t), h'(t))$$

this is the <u>trajectory</u> of the geodesic field G which <u>takes the value</u> (q, av) at $t=0$.

So $(h(t), h'(t)) = \gamma(t, q, av)$

$$\| $$

$$\gamma(at, q, v) \qquad \square.$$

<u>Prop. 2.7</u>: Given $p \in M$, $\exists\ \varepsilon > 0$ and a neighborhood V of p and a C^∞ map

$$\gamma :]-2, 2[\times V \times B_\varepsilon(0) \longrightarrow TM$$

s.t. $t \longmapsto \gamma(t, q, v)$ is the unique trajectory of G taking the value (q, v) at 0.

Proof: $\exists\ \delta,\ \varepsilon_1 > 0$ and

$\gamma_1 : \,]-\delta, \delta[\, \times V \times B_{\varepsilon_1}(0) \to TM$

as above

~~define~~ note: $\gamma_1(at, q, v) = \gamma_1(t, q, av)$

take $a = \frac{2}{\delta}$.

define $\gamma_*(t, q, v) := \gamma_1(\frac{2}{\delta} t, q, v)$

~~then~~ $\varepsilon = \frac{\delta}{2}\varepsilon_1$ $\square$.

Prop: Given $p \in M$, $\exists\ \delta > 0$ and a neighborhood V of p and a C^∞

$\gamma : \,]-\delta, \delta[\, \times V \times B_2(0) \to TM$

s.t. $t \longmapsto \gamma(t, q, v)$ is the unique trajectory of G taking the value (q, v) at 0.

Proof: $\exists\ \delta_1, \varepsilon > 0$ and

$\gamma_1 : \,]-\delta_1, \delta_1[\, \times V \times B_\varepsilon(0) \to TM$

as above. $\gamma_1(at, q, v) = \gamma_1(t, q, av)$

take $a = \frac{2}{\varepsilon}$ define $\gamma(t, q, v) = \gamma_1(at, q, v)$

then $\delta = \frac{\varepsilon}{2}\delta_1$ $\square$

Definition: The exponential map is the map $\mathcal{V}_\varepsilon \longrightarrow TM$.

$$(q,v) \longmapsto \gamma(1,q,v)$$

where γ, ε, $\mathcal{V}$ are as in Prop. 2.7.

Note $\gamma(1,q,v) = \gamma\left(|v|, q, \frac{v}{|v|}\right)$