Homework 1: Chapter 0

(2) M differentiable manifold. Show TM is orientable.

Proof: $\{(U_\alpha, \mathbf{x}_\alpha)\}$ atlas for M.

$\rightsquigarrow$ atlas for TM:
$$\left.\{(U_\alpha \times \mathbb{R}^n, (\mathbf{x}_\alpha, d\mathbf{x}_\alpha))\}\right\} \text{ is an orientation}$$

on $U_\alpha \cap U_\beta$: change of coordinate

$$\mathbf{x}_\alpha^{-1} \circ \mathbf{x}_\beta : U_\beta \cap \mathbf{x}_\beta^{-1}(U_\alpha) \to \mathbb{R}^n$$

coordinates on $U_\alpha \times \mathbb{R}^n$: $\underbrace{(x_1, \dots, x_n, u_1, \dots, u_n)}$ $\to \sum u_i \frac{\partial}{\partial x_i}$

change of coordinates for TM:

(on $U_\beta \times \mathbb{R}^n$: $(y_1, \dots, y_n, \underbrace{v_1, \dots, v_n}_{\sum v_i \frac{\partial}{\partial y_i}})$)

$$(\mathbf{x}_\alpha^{-1} \circ \mathbf{x}_\beta, d(\mathbf{x}_\alpha^{-1} \circ \mathbf{x}_\beta))$$

the differential of the change of coordinates:

$$\left(\begin{array}{ccc|ccc}
\frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} & \frac{\partial x_1}{\partial v_1} & \cdots & \frac{\partial x_1}{\partial v_n} \\
\vdots & & & & \vdots & \\
\hline
\frac{\partial u_1}{\partial y_1} & \cdots & \frac{\partial u_1}{\partial y_n} & \frac{\partial u_1}{\partial v_1} & \cdots & \frac{\partial u_1}{\partial v_n} \\
\vdots & & & \vdots & &
\end{array}\right)$$

Note: $\frac{\partial u_i}{\partial v_j} = \frac{\partial x_i}{\partial y_j}$ because $\left(\frac{\partial x_i}{\partial y_j}\right)$ is the change of coordinate matrix from (x_i) to (y_i), the change of coordinates is linear so its derivative is itself.

x_i is independent of $v_1, \dots, v_n$, so

$$\frac{\partial x_i}{\partial v_j} = 0 \quad \forall i, j$$

So det of matrix is the square of $\det\left(\frac{\partial x_i}{\partial y_j}\right)$ which is >0.

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(3) (a) $S \subset \mathbb{R}^3$.

Assume S is orientable

$$S = \bigcup \mathbf{x}_\alpha(U_\alpha), \quad \mathbf{x}_\alpha : U_\alpha \longmapsto S$$

(x, y) coordinates.

$\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}$ basis of TS on $\mathbf{x}_\alpha(U_\alpha)$.

on U_α : $N_\alpha := \dfrac{\frac{\partial}{\partial x} \times \frac{\partial}{\partial y}}{\left|\frac{\partial}{\partial x} \times \frac{\partial}{\partial y}\right|}$

Show $N_\alpha\big|_{U_\alpha \cap U_\beta} = N_\beta\big|_{U_\alpha \cap U_\beta}$.

on U_β: (a, b). coordinates.

$$\frac{\partial}{\partial x} \times \frac{\partial}{\partial y} = \det\left(\frac{\partial(a,b)}{\partial(x,y)}\right) \cdot \frac{\partial}{\partial a} \times \frac{\partial}{\partial b}$$

$\Longrightarrow$ $N_\alpha = N_\beta$ on $U_\alpha \cap U_\beta$.

because $\det\left(\frac{\partial(a,b)}{\partial(x,y)}\right) > 0$.

Now assume $\exists$ unit normal

$$N : S \longrightarrow \mathbb{R}^3.$$

Let $\{(V_\alpha, x_\alpha)\}$ be an atlas.

$\forall \alpha$ $N_\alpha :=$ as before

~~then~~ assume each V_α is connected.

then $\forall \alpha$, $N\big|_{V_\alpha} = \pm N_\alpha$

if $N\big|_{V_\alpha} = N_\alpha$ do nothing

$N\big|_{V_\alpha} = -N_\alpha$ switch the coordinates on V_α to get $N\big|_{V_\alpha} = N_\alpha$

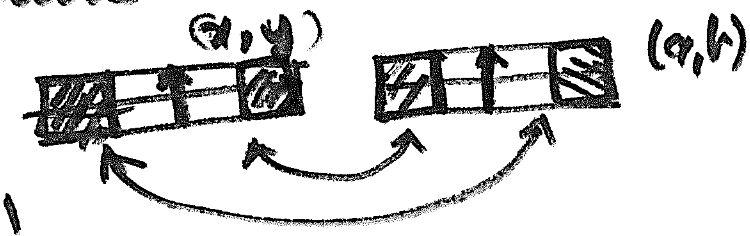
prove this modified atlas is an orientation: use the formula:

$$\frac{\partial}{\partial x} \times \frac{\partial}{\partial y} = \det\left(\frac{\partial(a,b)}{\partial(x,y)}\right) \frac{\partial}{\partial a} \times \frac{\partial}{\partial b}$$

$$N_\alpha\big|_{U_\alpha \cap U_\beta} = N\big|_{U_\alpha \cap U_\beta} = N_\beta\big|_{U_\alpha \cap U_\beta}$$

$$\Rightarrow \quad \det\left(\frac{\partial(a,b)}{\partial(x,y)}\right) > 0$$

(b) the Möbius band is the union of two open sets:



$-1 < x < 1 \qquad -1 < a < 1$

$-1 < y < 1 \qquad -1 < y < 1$

$$(*) \begin{cases} x \longleftrightarrow a + 3/2 \text{ on } & x \in (1/2, 1) \\ y \longleftrightarrow b & y \in (-1, 1) \\ \hline x \longleftrightarrow a - 3/2 & x \in (-1, -1/2) \\ y \longleftrightarrow -b & y \in (-1, 1) \end{cases}$$

If we had a unit normal, it would be obtained from glueing unit normals on the two pieces.

The differential of the coordinate change in the second open set above

has negative determinant and we cannot glue the normals.

$\mathfrak{X}_1 : U_1 \longrightarrow \mathbb{R}^3$

$\mathfrak{X}_2 : U_2 \longrightarrow \mathbb{R}^3.$

$\mathfrak{X}_2^{-1} \circ \mathfrak{X}_1 : \boxed{//} \underset{(a,b)}{\cup} \boxed{\backslash\backslash} \longrightarrow \mathbb{R}^2.$

is given by the maps $(*)$.

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(5) $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$

$(x,y,z) \longmapsto (x^2-y^2, xy, xz, yz)$

$$\begin{tikzcd} S^2 \arrow[r, hook, "i"] \arrow[d, "\pi"'] \arrow[dr, "\varphi"] & \mathbb{R}^3 \arrow[d, "F"] \\ \mathbb{P}^2(\mathbb{R}) \arrow[r, hook, "?", "\tilde{\varphi}"'] & \mathbb{R}^4 \end{tikzcd}$$

(a) $\tilde{\varphi}$ is an immersion.

<u>Proof</u>: <u>Note</u>: equivalent to φ is an immersion.

because π is a local diffeo. so at any $p \in S^2$ $\quad T_p S^2 \xrightarrow{\cong} T_{\pi(p)} \mathbb{P}^2(\mathbb{R})$

$$d\varphi : T_p S^2 \longrightarrow T_{\varphi(p)} \mathbb{R}^4$$

dF has matrix $\begin{pmatrix} 2x & -2y & 0 \\ y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{pmatrix}$

3×3 minors: $2x^3 + 2xy^2 = x(2x^2+2y^2)$

$2xyz$

$-2x^2 z + 2y^2 z = z(2x^2+2y^2)$

$-2xyz$

if matrix does not have rank 3, ~~either~~ then $x=y=0$ ~~or $x=z=0$~~

On the sphere this means:

$x=y=0,\ z=\pm 1$ ~~or $x=z=0,\ y=\pm 1$~~

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \end{pmatrix} \qquad \cancel{\begin{pmatrix} 0 & \mp 2 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}}$$

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$(x,y) \longmapsto \big((r\cos y + a)\cos x,\ (r\cos y + a)\sin x,\ r\sin y\big).$$

the antipodal map acts as

$i : (x,y) \longmapsto (x+\pi, -y)$ on $S^1 \times S^1 = T^2$

$T^2/i = K$ the Klein bottle

We represent T^2 as $\mathbb{R}^2 / 4\pi\mathbb{Z} \oplus 2\pi\mathbb{Z}$.

then $i : (x,y) \longmapsto (x+2\pi, -y)$

Show that G factors through

$\mathbb{R}^2 / 4\pi\mathbb{Z} \oplus 2\pi\mathbb{Z} = T^2$ and induces an embedding of T^2.

Show $G(x,y) = G(x+2\pi, -y)$. ✓

Show if $G(x,y) = G(c,d)$, then

either $(x,y) = (c,d)$ or $(x,y) = (c+2\pi, -d)$

$(r\cos y + a)\cos x = (r\cos d + a)\cos c$

$(r\cos y + a)\sin x = (r\cos d + a)\sin c$

$r\sin y \cos\frac{x}{2} = r\sin d \cos\frac{c}{2}$

$r\sin y \sin\frac{x}{2} = r\sin d \sin\frac{c}{2}.$

if $\cos x \neq 0$, then $\tan x = \tan c$

$\Rightarrow x - c = 0, \pi, 2\pi, 3\pi$

if $\sin x \neq 0$, then $\cot x = \cot c$

$\Rightarrow x - c = 0, \pi, 2\pi, 3\pi$

$\Rightarrow \cos\frac{x}{2} = \pm \cos\frac{c}{2}$ or $\pm \sin\frac{c}{2}$.

the second pair of equations gives:

$\left\{\begin{array}{l} \cot\frac{x}{2} = \cot\frac{c}{2} \quad \Rightarrow x - c = 0, 2\pi. \\ \text{or } \tan\frac{x}{2} = \tan\frac{c}{2}. \end{array}\right.$

if $\sin y \neq 0$

if $\sin y = 0$, then the two equations give $\sin d = 0$.

$\Rightarrow$ ~~d = y = d or y =~~

y & d are 0 or π

the first two equations give

$r\cos y + a = r\cos d + a$

$\Rightarrow \cos y = \cos d \Rightarrow y = d = 0$ or π.

if $x - c = 0$ or 2π, go back to equations 3,4: if $x = c$, then $\sin y = \sin d$.

if $x = c + 2\pi$, then $\sin y = -\sin d$.

Kernel: $\begin{pmatrix}0\\0\\1\end{pmatrix}$

tangent plane to S^2: (x,y) plane, does not contain $\begin{pmatrix}0\\0\\1\end{pmatrix}$

~~Kernel~~ $\cancel{\begin{pmatrix}1\\1\\1\end{pmatrix}}$

~~tangent plane to~~ $\cancel{S^2}$ ~~is the~~ $\cancel{(x,z)}$ ~~plane, does not contain~~ $\cancel{\begin{pmatrix}1\\1\\1\end{pmatrix}}$

So $d\varphi = dF \circ di$ is always injective on T_pS^2 $(\forall p)$.

(b) $\tilde{\varphi}$ is injective

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(6) $G : \mathbb{R}^2 \to \mathbb{R}^4$

$$(x,y) \longmapsto \left((r\cos y + a)\cos x,\ (r\cos y + a)\sin x,\ r\sin y\cos\frac{x}{2},\ r\sin y\sin\frac{x}{2}\right)$$

induces an embedding of the Klein bottle.

This was defined as the quotient of $T^2 := S^1 \times S^1 \subset \mathbb{R}^3$ by the antipodal map of $\mathbb{R}^3$.

T^2 is the image of the map