(10) $\pi : M \longrightarrow M/G$. open.

<u>show</u>: the topology (quotient top.) on M/G is Hausdorff iff
$\forall\ p_1, p_2 \in M$
$\exists$ neighborhoods U_1 of p_1, U_2 of p_2
s.t. $U_1 \cap gU_2 = \phi \quad \forall\ g \in G$.

<u>Proof</u>: ($\Leftarrow$) for any $q_1, q_2 \in M/G$
choose $p_1, p_2 \in M$ s.t. $\pi(p_1) = q_1, \pi(p_2) = q_2$
choose U_1, U_2 as above
$V_1 := \pi(U_1) \qquad V_2 := \pi(U_2)$. open
verify $V_1 \cap V_2 = \phi \qquad (q_1 \in V_1, q_2 \in V_2)$

($\Rightarrow$) assume M/G Hausdorff
$\forall\ p_1, p_2 \in M$, $\exists\ V_1 \ni \pi(p_1)$
$V_2 \ni \pi(p_2)$
s.t. $V_1 \cap V_2 = \phi$
$U_1 := \pi^{-1}(V_1) \qquad U_2 := \pi^{-1}(V_2)$
$U_1 \cap U_2 = \phi = \pi^{-1}(V_1 \cap V_2) = \pi^{-1}(\phi)$

$\forall\, g \in G \quad gU_1 = U_1 \;,\; gU_2 = U_2$

$\Rightarrow \quad U_1 \cap gU_2 = \phi \qquad \square.$

(12). M non-orientable

$p \in M \qquad \{\text{bases of } T_pM\}$

$\{v_1, \dots, v_n\}$, $\{w_1, \dots, w_n\}$ bases.

$\exists!\; g \in GL_n(\mathbb{R})$. s.t. $\forall\, i$, $gv_i = w_i$

Choose one basis $\{v_1, \dots, v_n\} =: B$.

all others are elements of

$$GL_n(\mathbb{R}) \cdot B.$$

(this is an example of a principal homogeneous space)

$$\underbrace{\boxed{GL_n(\mathbb{R})}}_{\text{Lie group.}} = \underbrace{\boxed{GL_n^+(\mathbb{R})}}_{\text{positive det}} \coprod \underbrace{GL_n^-(\mathbb{R})}_{\text{negative det.}}$$

the connected component of $1 \in GL_n(\mathbb{R})$

$$\tilde{M} := \{(p, B) : p \in M,\ B \text{ a basis of } T_pM\}.$$

Show $\tilde{M}$ is a manifold..

$GL_n(\mathbb{R})$ acts on $\tilde{M}$

$$M = \tilde{M} / GL_n(\mathbb{R})$$

$$GL_n^+(\mathbb{R}) \subset GL_n(\mathbb{R})$$

acts on $\tilde{M}$

$$\bar{M} := \tilde{M} / GL_n^+(\mathbb{R}) \xrightarrow{2\text{-to-}1} M$$

(a) $\{(U_\alpha, \varphi_\alpha)\}$ differentiable structure on M.

Define a differentiable structure on $\bar{M}$:

$\varphi_\alpha : U_\alpha \longrightarrow M$. $\qquad U_\alpha \subset \mathbb{R}^n$

$$\bar{\varphi}_\alpha : U_\alpha \longrightarrow \bar{M}$$

$$(x_1, \dots, x_n) \longmapsto \left(\varphi_\alpha(x_1, \dots, x_n), GL_n^+(\mathbb{R})\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}\right)$$

$$\bar{\varphi}_\alpha' : U_\alpha \longrightarrow \bar{M}$$

$$(x_1, \dots, x_n) \longmapsto \left(\varphi_\alpha(x_1, \dots, x_n), GL_n^+(\mathbb{R})\left\{-\frac{\partial}{\partial x_1}, \dots, +\frac{\partial}{\partial x_n}\right\}\right)$$

$\{(V_\alpha, \bar\varphi_\alpha), (V_\alpha, \bar\varphi'_\alpha)\}$ covers $\bar M$ and is a differentiable structure.

e.g.: $\bar\varphi_\beta^{-1} \circ \bar\varphi_\alpha = \varphi_\beta^{-1} \circ \varphi_\alpha$ is ~~differentiable.~~

$\bar\varphi_\beta'^{-1} \circ \bar\varphi_\alpha$ ~~#~~: no overlap if $\varphi_\beta, \varphi_\alpha$ overlap.

(b) $\pi : \bar M \longrightarrow M$. surjective by def.

$(p, O_p) \longmapsto p$.

differentiable essentially by def.

$p \in M$. $\quad p \in \varphi_\alpha(V_\alpha)$.

$$\pi^{-1}(\varphi_\alpha(V_\alpha)) = \bar\varphi_\alpha(V_\alpha) \amalg \bar\varphi'_\alpha(V_\alpha).$$

diffeo. $\cong \qquad \cong$

$$\varphi_\alpha(V_\alpha) \qquad \varphi_\alpha(V_\alpha)$$

So $\pi : \bar M \longrightarrow M$ is a double covering and a local diffeo.

(c) $\overline{\mathbb{P}^2} \longrightarrow \mathbb{P}^2 \longleftarrow S^2$

$$\bar\varphi_1 : U_1 \longrightarrow S^2 \qquad U_1 \subset \mathbb{R}^2$$

$$(x,y) \longmapsto (x,y,\sqrt{1-x^2-y^2}) \qquad B_1(0)$$

image = upper hemisphere

$$\bar\varphi_1' : U_1 \longrightarrow S^2$$

$$(x,y) \longmapsto (x,y,-\sqrt{1-x^2-y^2})$$

$\bar\varphi_2$, $\bar\varphi_2'$, $\bar\varphi_3$, $\bar\varphi_3'$ similar.

define map $S^2 \longrightarrow \overline{\mathbb{P}}^2$.

$$(x,y,\sqrt{1-x^2-y^2}) \longmapsto \left([x,y,\sqrt{1-x^2-y^2}],\left[\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right]\right)$$

$$(x,y,-\sqrt{1-x^2-y^2}) \longmapsto \left([x,y,\sqrt{1-x^2-y^2}],\left[-\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right]\right)$$

$$\bar\varphi_2 : U_2 \longrightarrow S^2$$

$$(y,z) \longmapsto (\sqrt{1-y^2-z^2},y,z).$$

map from $S^2 \to \overline{\mathbb{P}}^2$

$$(\sqrt{1-y^2-z^2},y,z) \longmapsto \left([\sqrt{1-y^2-z^2},y,z],\left[\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right]\right)$$

$$(x,y) \longmapsto \left(\left[x, y, \sqrt{1-x^2-y^2}\right], \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]\right)$$

$$\longmapsto (y, z) \quad \text{where.}$$

$$y = y \qquad z = \sqrt{1-x^2-y^2}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z}.$$

So $$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y}.$$

$$x = \sqrt{1-y^2-z^2} \qquad \frac{\partial x}{\partial z} = -\frac{z}{\sqrt{1-y^2-z^2}}$$

$$= -\frac{z}{x}$$

$$\frac{\partial}{\partial z} = -\frac{z}{x}\frac{\partial}{\partial x} - \frac{y}{x}\frac{\partial}{\partial y}$$

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] \longrightarrow \left[\frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$$

$$\left[\frac{\partial}{\partial y}, -\frac{z}{x}\frac{\partial}{\partial x} - \frac{y}{x}\frac{\partial}{\partial y}\right]$$

$$\begin{pmatrix} 0 & 1 \\ -\frac{z}{x} & -\frac{z}{y} \end{pmatrix} \qquad \det = \frac{z}{x} > 0 \text{ on } \overline{\varphi_1}, \overline{\varphi_2}$$

So the maps on hemispheres glue to give $S^2 \xrightarrow{\alpha} \overline{\mathbb{P}^2}$.

$$\begin{tikzcd} S^2 \arrow[r, "\alpha"] \arrow[dr] & \overline{\mathbb{P}^2} \arrow[d] \\ & \mathbb{P}^2 \end{tikzcd}$$

commutative

$\Rightarrow$ local diffeo. because the two projections are local diffeo.

Verify: ~~α~~ is a bijection.

<u>surjectivity</u>: both orientations at each point of $\mathbb{P}^2$ are in the image of α. (by construction).

If $a, b \in S^2$ s.t. $\alpha(a) = \alpha(b)$, ~~then~~ a, b ~~$\alpha(a), \alpha(b)$~~ map to the same point of $\mathbb{P}^2$, so $a = \pm b$

a and $-a$ go to different points of $\overline{\mathbb{P}^2}$.